ARE 521 QUANTITATIVE TECHNIQUES

Introduction to LP and Simplex Method

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Linear Programming

- Mathematical programming technique for solving optimization problem (either maximization or minimization problem)

- Constraints and the functions to be maximized or minimized are linear, thus can be represented as straight lines.

General LP Problem can be expressed as

$$Max(or Min) Z = \sum_{J=1}^{n} c_j x_j$$

Subject to

$$\sum_{J=1}^{n} a_{ij} x_j (\leq, =, \geq) b_i, (i = 1, 2, \ldots\ldots.m)$$

$$x_j \geq 0, (j = 1, 2, \ldots\ldots\ldots n)$$

- Consider following LP Problem
- Primal   Dual :

max $Z = c'x$

subject to

$$Ax \le r$$
$$x \ge 0$$

min $Z^* = r'y$

subject to

$$A'y \ge c$$
$$y \ge 0$$

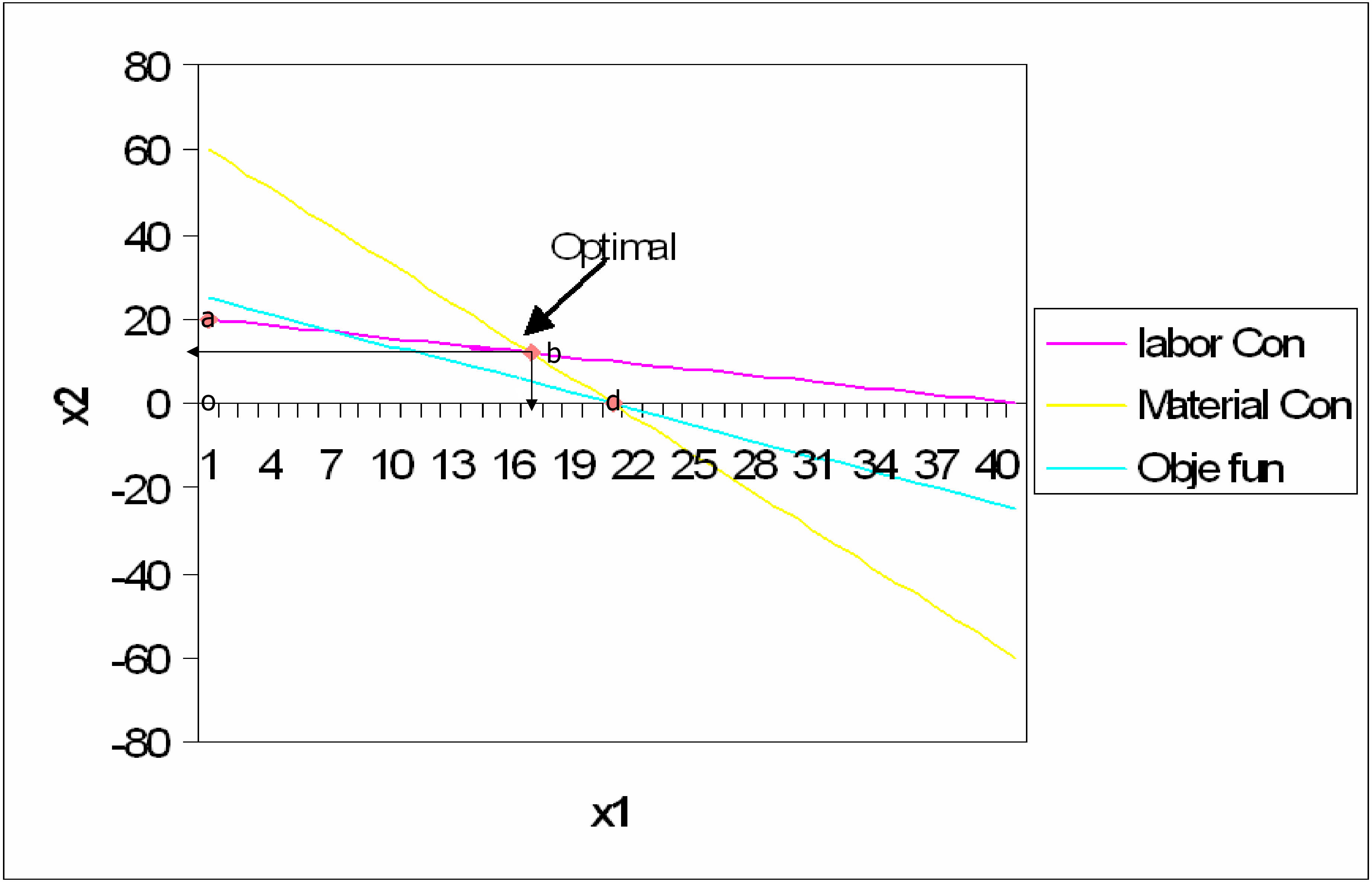
$$Max(Z) = 100x_1 + 80x_2$$

$$subjeto$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 80 \\ 60 \end{bmatrix}$$

$$x_1, x_1 \ge 0$$

Graphical Solution to LP



Gauss Jordan Elimination Method

- A method for determining the inverse of a matrix

- Can be used to solve system of linear equations

- Given a set of linear equations, an equivalent set of equations with same solution set can be generated by replacing one of the original equations with another equation. The replacement equation is computed by adding to the replaced equation some multiple of one of the other equation in the original set.

$$\left.\begin{array}{l} 2x_1 + 4x_2 = 80 \text{ ---1} \\ 3x_1 + 1x_2 = 60 \text{ ---2} \end{array}\right\} \longrightarrow \begin{array}{l} 1x_1 + 0x_2 = A \\ 0x_1 + 1x_2 = B \end{array}$$

Solving a simultaneous equation system using Gauss-Jordan Procedure

$2x_1 + 4x_2 = 80$ ---1
$3x_1 + 1x_2 = 60$ ---2

$1x_1 + 0x_2 = A$
$0x_1 + 1x_2 = B$

$1x_1 + 2x_2 = 40$ ---1a — By dividing 1 by 2
$3x_1 + 1x_2 = 60$ ---2

$-3x_1 - 6x_2 = -120$ ---1a — multiplying 1a by -3
$3x_1 + 1x_2 = 60$ --- 2
$0x_1 - 5x_2 = -60$ --- 2a

$1x_1 + 2x_2 = 40$ ---1a
$0x_1 - 5x_2 = -60$ --- 2a — divided 2a by -5

$1x_1 + 2x_2 = 40$ ---1a
$0x_1 + 1x_2 = 12$ --- 3

$0x_1 - 2x_2 = -24$ --- 3a — multiplying 3 by -2
$1x_1 + 2x_2 = 40$ --- 1a
$x_1 + 0x_2 = 16$

$1x_1 + 0x_2 = 16$ ---2
$0x_1 + 1x_2 = 12$ --- 3 a

$$\left[\begin{matrix}2 & 4\\3 & 1\end{matrix}\right]\left[\begin{matrix}1 & 0\\0 & 1\end{matrix}\right]$$

Divided row1 by 2

$$\left[\begin{matrix}1 & 2\\3 & 1\end{matrix}\right]\left[\begin{matrix}1/2 & 0\\0 & 1\end{matrix}\right]$$

Multiplied row 1 by -3 and add the result to 2nd row

$$\left[\begin{matrix}1 & 2\\0 & -5\end{matrix}\right]\left[\begin{matrix}1/2 & 0\\-3/2 & 1\end{matrix}\right]$$

Divided 2 row by -5

$$\left[\begin{matrix}1 & 2\\0 & 1\end{matrix}\right]\left[\begin{matrix}1/2 & 0\\3/10 & -1/5\end{matrix}\right]$$

Multiplied row 2 by -2 and add to the 1 row

$$\left[\begin{matrix}1 & 0\\0 & 1\end{matrix}\right]\underbrace{\left[\begin{matrix}-1/10 & 2/5\\3/10 & -1/5\end{matrix}\right]}_{}$$

A inverse

$$\left[\begin{matrix}-1/10 & 2/5\\3/10 & -1/5\end{matrix}\right]\left[\begin{matrix}80\\60\end{matrix}\right]=\left[\begin{matrix}16\\12\end{matrix}\right]$$

Inverse of a matrix by Using Gauss Jordan Procedure

Cramer's rule

$$AX = b$$

$$X_i = \frac{|A_i|}{|A|}$$

Where A_i denotes a new determinant that is obtained after replacing column i of A with b

$$Max(Z) = 100x_1 + 80x_2$$
$$subjeto$$
$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 80 \\ 60 \end{bmatrix}$$
$$x_1, x_1 \geq 0$$

$$x_1 = \frac{\begin{vmatrix} 80 & 4 \\ 60 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}} = \frac{-160}{-10} = 16 \qquad x_2 = \frac{\begin{vmatrix} 2 & 80 \\ 3 & 60 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}} = \frac{-120}{-10} = 12$$

X_1= 16 x_2=12

Z= 16 * 100 + 12* 80
 = 1600 + 960
 = 2560

Linear programming and Simplex Method

- Developed by George Dantzig (1947)

- Based on Gauss Jordan elimination method

  - a set of simultaneous equations is solved through the matrix inverse procedure

- Suitable for solving linear programming problems involving large number of variables and constraints

- Iterative procedure: beginning at some initial feasible solution (a corner point of the feasible set, S, usually the origin), each iteration brings us to another corner point of 'S' with an improved (but certainly no worse) value of the objective function

- Standard linear programming problem is presented in TABLEAU form.

$$\text{Max } Z = 100x_1 + 80x_2 \quad \text{(Profit)}$$

subject to

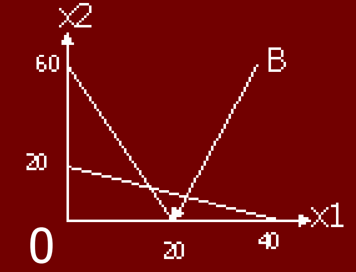
$$2x_1 + 4x_2 \leq 80 \text{ --- (labor)}$$

$$3x_1 + 1x_2 \leq 60 \text{ --- (Material)}$$

At, o

$X_1, x_2 = 0$

$s_1 = 80 \quad s_2 = 60$



Primary requirement of the simplex method :

- Convert inequality constraints to equalities
  - s_1 and s_2 are called as slack variables

$$\text{Max } Z = 100x_1 + 80x_2 + 0s_1 + 0s_2 \quad \text{(Profit)}$$

subject to

$$2x_1 + 4x_2 + 1\,s_1 + 0\,s_2 = 80 \text{ --- (labor)}$$

$$3x_1 + 1x_2 + 0\,s_1 + 1\,s_2 = 60 \text{ --- (Material)}$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Simplex Tableau

C_j			C_1	C_2	C_3	C_4
C_b	Basis	bi*	x1	x2	s1	s2
			a_{ij}			
	zj					
	cj-zj					

C_j : profit or cost coefficients (C_1, C_2 , C_3, C_4)

C_b : The profit or cost coefficients of the objective function for variable in the basis solution

Basis : the variables currently in the solution set . Combination of variables and associated values is referred to as a basic feasible solution

bi* : initially the values on the right-hand side of the constraints

Cj-Zj : criterion equation which guides the solution

$\text{Max } Z = 100x_1 + 80x_2 + 0s_1 + 0s_2$ (Profit)

subject to

$2x_1 + 4x_2 + 1\,s_1 + 0\,s_2 \leq 80$ --- (labor)

$3x_1 + 1x_2 + 0\,s_1 + 1\,s_2 \leq 60$ --- (Material)

$x_1 , x_2 , s_1 , s_2 \geq 0$

C_j			100	80	0	0
C_b	Basis	bi*	x1	x2	s1	s2
0	S_1	80	2	4	1	0
0	S_2	60	3	1	0	1
	Zj	0	0	0	0	0
	Cj-Zj	0	100	80	0	0

Initial basic feasible solution (Initial Tableau)

C_j			100	80	0	0	
C_b	Basis	bi*	x1	x2	s1	s2	
0	S1	80	2	4	1	0	
0	S2	60	3	1	0	1	
	Zj	0	0	0	0	0	
	Cj-Zj		100	80	0	0	

Pivot column (largest increase in profit per unit)

C_j			100	80	0	0	
C_b	Basis	bi*	x1	x2	s1	s2	
0	S1	80	2	4	1	0	40
0	S2	60	3	1	0	1	20
	Zj	0	0	0	0	0	
	Cj-Zj		100	80	0	0	

Pivot row : minimum ratio of b_i^*/ a_{ik} for $a_{ik} > 0$

Pivot element

X1 enter to the basis and S2 leave from the basis

New Tableau

New Pivot row (NPR)= $\dfrac{\text{old pivot row values}}{\text{Pivot element}}$ = $\dfrac{60 \quad 3 \quad 1 \quad 0 \quad 1}{3}$

New Row Value = Old row value - (Corresponding coefficient in the Pivot Column * new pivot row value)

New pivot row

Old row

C_j			100	80	0	0	
C_b	Basis	bi*	x1	x2	s1	s2	
0	S1	40	0	3 1/3	1	- 2/3	
100	x1	20	1	1/3	0	1/3	
	Zj						
	Cj-Zj						

New row value

Old row value

Coefficient of old row in Pivot Column

New pivot row values

Columns	ORV	-	Co in PC	X	NPRV	=	NRV
bi*	80	-	2	X	20	=	40
X1	2	-	2	X	1	=	0
X2	4	-	2	X	1/3	=	3 1/3
S1	1	-	2	X	0	=	1
S2	0	-	2	X	1/3	=	- 2/3

New row values

New Tableau

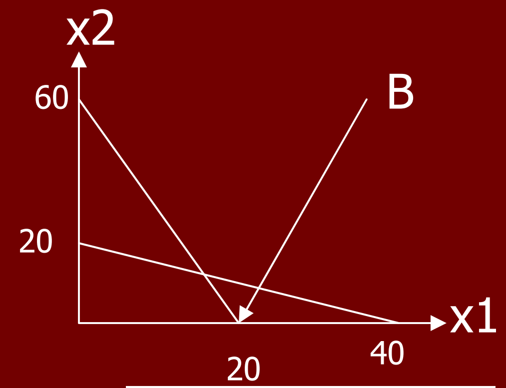
C_j			100	80	0	0	
C_b	Basis	bi*	x1	x2	s1	s2	
0	S1	40	0	3 1/3	1	- 2/3	
100	x1	20	1	1/3	0	1/3	
	Zj	2000	100	33 1/3	0	33 1/3	
	Cj-Zj		0	46 2/3	0	-33 1/3	

$Z_{bi*} = 2000$

Activity in the basis X1 (real activity) = 20

s1 (disposal activity= 40)

Positive Cj-Zj = 46 2/3 (can improve the basis)



At, B

$X_1 = 20$, $x_2 = 0$

$s_1 = 40$ $s_2 = 0$

C_j			100	80	0	0	
C_b	Basis	bi*	x1	x2	s1	s2	
0	S1	40	0	3 1/3	1	- 2/3	12
100	x1	20	1	1/3	0	1/3	60
	Zj	2000	100	33 1/3	0	33 1/3	
	Cj-Zj		0	46 2/3	0	-33 1/3	

Pivot row

Pivot column

- X_2 enters to the basis and S_1 leave from the basis

- New pivot row

  divided pivot row by pivot element i.e. 3 1/3

$$\frac{40 \quad 0 \quad 3\,1/3 \quad 1 \quad -2/3}{3\,1/3} = 12 \quad 0 \quad 1 \quad 3/10 \quad -1/5$$

- calculate new row

Columans	ORV	-	Co in PC	X	NPRV	=	NRV
bi*	**20**	-	1/3	X	12	=	16
X1	1	-	1/3	X	0	=	1
X2	1/3	-	1/3	X	1	=	0
S1	**0**	-	1/3	X	0.3	=	-0.1
S2	**1/3**	-	1/3	X	- 1/5	=	2/5

C_j			**100**	**80**	**0**	**0**
C_b	**Basis**	**bi***	**x1**	**x2**	**s1**	**s2**
80	**x2**	**12**	**0**	**1**	**0.30**	**- 1/5**
100	**x1**	**16**	**1**	**0**	**-0.1**	**2/5**
	Zj	**2560**	**100**	**80**	**14**	**24**
	Cj-Zj		**0**	**0**	**-14**	**-24**

Total profits

Opportunity cost

Shadow prices

(Lagrangian multipliers)

- Economic Interpretation of Optimal tableau
- At optimum $x1= 16$ $x2= 12$
- Total profits (Z) = 16*100 + 12 * 80 =2560
- Resource are binding
  - Marginal value of the labor resource is 14
  - Marginal value of the material resource is 24
- Euler's Theorem

  TVP (Total value product) = $MVP_i * X_i$

  TVP= 14 * 80 + 24*60 = 2560

Linear programming and Simplex Method

	Constr level	Corn	Oats	Soy	Wheat	Con type	RHS
Activity level							
GM	92	30	10	40	12		
Land	4	1	1	1	1	≤	100
Mlab	1.5	0	1	0	0.5	≤	100
Jlab	3	1	0	2	0	≤	80

Max $Z = 30x_1 + 10x_2 + 40x_3 + 12x_4 + 0s_1 + 0s_2 + 0s_3$ (Profit)

subject to

$1x_1 + 1x_2 + 1x_3 + 1x_4 + 1s_1 + 0s_2 + 0s_3 = 100$ --- (Land)

$0x_1 + 1x_2 + 0x_3 + .5x_4 + 0s_1 + 1s_2 + 0s_3 = 100$ --- (March Labor)

$1x_1 + 0x_2 + 2x_3 + 0x_4 + 0s_1 + 0s_2 + 1s_3 = 80$ --- (January Labor)

$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0$

Simplex Tableau

Pivot Column

C_j			30	10	40	12	0	0	0	
C_b	Basis	bi*	x1	x2	x3	x4	s1	s2	s3	
0	S1	100	1	1	1	1	1	0	0	100
0	S2	100	0	1	0	0.5	0	1	0	
0	S3	80	1	0	2	0	0	0	1	40
	Zj	0	0	0	0	0	0	0	0	
	Cj-Zj		30	10	40	12	0	0	0	

Pivot row

Columans	ORV	-	Co in PC	X	NPRV	=	NRV
bi*	100	-	1	X	40	=	60
X1	1	-	1	X	0.5	=	0.5
X2	1	-	1	X	0	=	1
X3	1	-	1	X	1	=	0
X4	1	-	1	X	0	=	1
S1	1	-	1	X	0	=	1
S2	0	-	1	X	0	=	0
S3	0	-	1	X	0.5	=	-0.5

Columans	ORV	-	Co in PC	X	NPRV	=	NRV
bi*	100	-	0	X	40	=	100
X1	0	-	0	X	0.5	=	0
X2	1	-	0	X	0	=	1
X3	0	-	0	X	1	=	0
X4	0.5	-	0	X	0	=	0.5
S1	0	-	0	X	0	=	0
S2	1	-	0	X	0	=	1
S3	0	-	0	X	0.5	=	0

C_j			30	10	40	12	0	0	0	
C_b	Basis	bi*	x1	x2	x3	x4	s1	s2	s3	
0	S1	60	0.5	1	0	1	1	0	-0.5	60
0	S2	100	0	1	0	0.5	0	1	0	200
40	X3	40	0.5	0	1	0	0	0	0.5	
	Zj	1600	20	0	40	0	0	0	20	
	Cj-Zj		10	10	0	12	0	0	-20	

Cj			30	10	40	12	0	0	0	
Cb	Basis	bi*	x1	x2	x3	x4	s1	s2	s3	
12	X4	60	0.5	1	0	1	1	0	-0.5	120
0	S2	70	-0.25	0.5	0	0	-0.5	1	0.25	-280
40	X3	40	0.5	0	1	0	0	0	0.5	80
	Zj	2320	26	12	40	12	12	0	14	
	Cj-Zj		4	-2	0	0	-12	0	-14	

Optimal Tableau

Cj			30	10	40	12	0	0	0
Cb	Basis	bi*	x1	x2	x3	x4	s1	s2	s3
12	X4	20	0	1	-1	1	1	0	-1
0	S2	90	0	0.5	0.5	0	-0.5	1	0.5
30	x1	80	1	0	2	0	0	0	1
	Zj	2640	30	12	48	12	12	0	18
	Cj-Zj		0	-2	-8	0	-12	0	-18

Opportunity cost

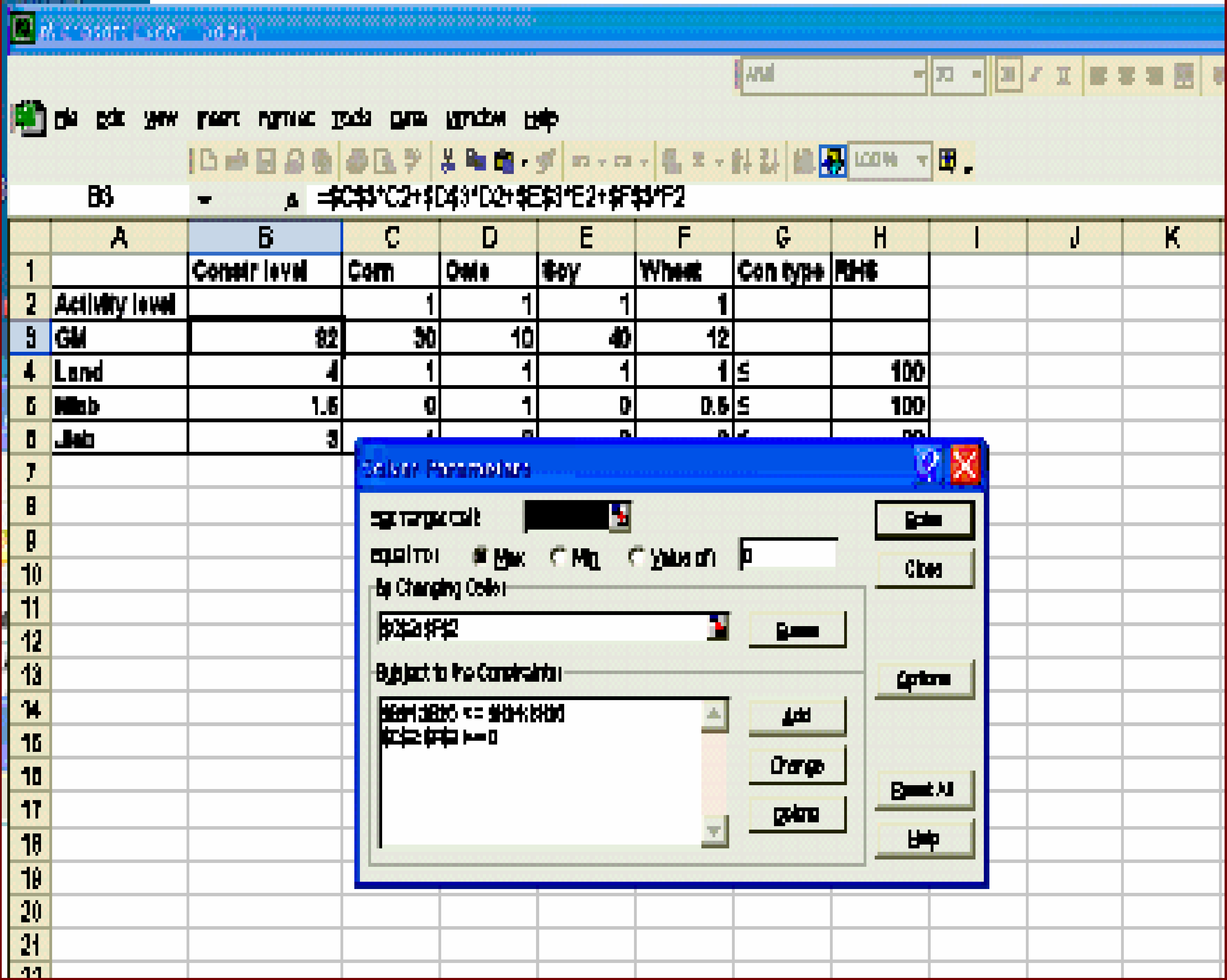
Shadow prices

Optimal objective functional value

Solving LP Problems directly in Excel

- The problem is entered in the spread sheet roughly as follows
- Note that there is an extra row (activity level row, called the changing cells in the Excel Solver terminology- (C2:F2) in this example
- Also there is an extra column (B3:B6) calculates the level of resource use associated with any given activity levels
- The value of cell B3 is of particular interest and maximized . It is the activity levels time the activity gross margin (=$C$3*C2+ $D$4*D2+ $E$4*E2+ $F$4*F2)

	Constr level	Corn	Oats	Soy	Wheat	Con type	RHS
Activity level		1	1	1	1		
GM	92	30	10	40	12		
Land	4	1	1	1	1	≤	100
Mlab	1.5	0	1	0	0.5	≤	100
Jlab	3	1	0	2	0	≤	80


B3
=$C$3*C2+$D$3*D2+$E$3*E2+$F$3*F2
A B C D E F G H I J K
Consir level
Corn
Soy
Wheat
Con type
RHS
Activity level 1 1 1 1
GM 92 30 10 40 12
Land 4 1 1 1 1 ≤ 100
1.5 0 1 0 0.5 ≤ 100
3
Solver Parameters
Max
Min
Value of:
By Changing Cells:
Subject to the Constraints:
Close
Options
Add
Change
Help

Microsoft Excel - Book1

=$C$3*C2+$D$3*D2+$E$3*E2+$F$3*F2

	A	B	C	D	E	F	G	H
1		Constr level	Corn	Oats	Soy	Wheat	Con type	RHS
2	Activity level		80	0	0	20		
3	GM	2640	30	10	40	12		
4	Land	100	1	1	1	1	≤	100
5	Mlab	10	0	1	0	0.5	≤	100
6	Jlab	80	1	0	2	0	≤	80

Solver Results

Solver found a solution. All constraints and optimality conditions are satisfied.

- (•) Keep Solver Solution
- ( ) Restore Original Values

Reports: Answer, Sensitivity, Limits

OK | Cancel | Save Scenario... | Help

Microsoft Excel 10.0 Answer Report
Worksheet: [Book1]Sheet1
Report Created: 8/20/2005 4:52:22 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
$B$3	GM Constr level	92	2640

Adjustable Cells

Cell	Name	Original Value	Final Value
$C$2	Activity level Corn	1	80
$D$2	Activity level Oats	1	0
$E$2	Activity level Soy	1	0
$F$2	Activity level Wheat	1	20

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$B$4	Land Constr level	100	$B$4<=$H$4	Binding	0
$B$5	Mlab Constr level	10	$B$5<=$H$5	Not Binding	90
$B$6	Jlab Constr level	80	$B$6<=$H$6	Binding	0
$C$2	Activity level Corn	80	$C$2>=0	Not Binding	80
$D$2	Activity level Oats	0	$D$2>=0	Binding	0
$E$2	Activity level Soy	0	$E$2>=0	Binding	0
$F$2	Activity level Wheat	20	$F$2>=0	Not Binding	20

Microsoft Excel 10.0 Sensitivity Report
Worksheet: [Book1]Sheet1
Report Created: 8/20/2005 4:52:22 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Gradient
$C$2	Activity level Corn	80	0
$D$2	Activity level Oats	0	-2.000000238
$E$2	Activity level Soy	0	-7.999999285
$F$2	Activity level Wheat	20	0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
$B$4	Land Constr level	100	12
$B$5	Mlab Constr level	10	0
$B$6	Jlab Constr level	80	18

Microsoft Excel 10.0 Limits Report
Worksheet: [Book1]Limits Report 1
Report Created: 8/20/2005 4:52:22 PM

Cell	Target Name	Value
$B$3	GM Constr level	2640

Cell	Adjustable Name	Value	Lower Limit	Target Result	Upper Limit	Target Result
$C$2	Activity level Corn	80	0	240	80	2640
$D$2	Activity level Oats	0	0	2640	0	2640
$E$2	Activity level Soy	0	0	2640	0	2640
$F$2	Activity level Wheat	20	0	2400	20	2640

General Algebraic Modeling System (GAMS)

- Designed to solve Linear, Nonlinear and Mixed integer optimization problems

- GAMS Language : formally similar to commonly used programming languages

- A GAMS Model is a collection of concise algebraic statements in the GAMS language

- http://www.gams.com/docs/intro.htm

- Structure of a GAMS Model

SETS declaration

assignment of members

DATA (PARAMETERS, TABLES, SCALARS)

declaration

assignment of Members

VARIABLES

declaration

assignment of type

EQUATIONS declaration

definition

MODEL and SOLVE statements

Program for the Optimization problem in GAMS (PRIMAL)

```
Option limrow=10, limcol=10;

SET  J    crops /1, 2, 3, 4/
       I   resources /land, Mlab , Jlab/;

PARAMETERS   c(J) grossMar / 1 30, 2 10, 3 40, 4 12/
                        b(I)   limit  / land 100, Mlab 100, Jlab 80/;

     TABLE  a(i,j) activity matrix
          1     2     3      4
land      1     1     1      1
Mlab     0     1     0     0.5
Jlab      1     0     2      0 ;

VARIABLE TGM;
positive variables x;

EQUATIONS
 obj
 constrt;

Obj .. TGM =e= sum (j, c(j)* x(j));
constrt(i).. sum(j, a(i,j)* x(j)) =L= b(i);

model farm/all/;
solve farm using lp maximizing TGM;
```

```
               S O L V E      S U M M A R Y

     MODEL    farm                 OBJECTIVE  TGM
     TYPE     LP                   DIRECTION  MAXIMIZE
     SOLVER   CPLEX                FROM LINE  30

**** SOLVER STATUS     1 NORMAL COMPLETION
**** MODEL STATUS      1 OPTIMAL
**** OBJECTIVE VALUE             2640.0000

                          LOWER     LEVEL     UPPER    MARGINAL

---- EQU obj                .         .         .        1.000

---- EQU constrt

        LOWER     LEVEL     UPPER    MARGINAL

land     -INF    100.000   100.000    12.000
Mlab     -INF     10.000   100.000      .
Jlab     -INF     80.000    80.000    18.000

                          LOWER     LEVEL     UPPER    MARGINAL

---- VAR TGM              -INF    2640.000    +INF        .

---- VAR x

     LOWER     LEVEL     UPPER    MARGINAL

1      .      80.000     +INF        .
2      .         .       +INF     -2.000
3      .         .       +INF     -8.000
4      .      20.000     +INF        .
```

Primal

$$\text{Max } Z = 30x_1 + 10x_2 + 40x_3 + 12x_4$$

$$\text{S.T} \quad x_1 + x_2 + x_3 + x_4 \le 100$$

$$x_2 + 0.5x_4 \le 100$$

$$x_1 + 2x_3 \le 80$$

$$x_1, x_2, x_3, x_4 \ge 0$$

Dual

$$\text{Min } C = 100Y_1 + 100Y_2 + 80Y_3$$

$$\text{S.T} \quad Y_1 + Y_3 \ge 30$$

$$Y_1 + Y_2 \ge 10$$

$$Y_1 + 2Y_3 \ge 40$$

$$Y_1 + .5Y_2 \ge 12$$

$$Y_1, Y_2, Y_3, Y_4 \ge 0$$

- Y_1 , Y_2 ,Y_3 are associated with Primal Land , Mlab and Jlab constraints consecutively
- Y_i , dual variable, can be interpreted as marginal value of each resource (Lagrangian multiplier)

Program for the Optimization problem in GAMS (DUAL)

```
Option limrow=10, limcol=10;

Set j crops /1,2,3,4/
     i resources /land, Mlab ,Jlab/;

Parameters c(j)   GM / 1 30, 2 10, 3 40, 4 12/
                b(i)   limit / land 100, Mlab 100, Jlab 80/;

Table a(j,i) activity matirx

        Land  Mlab  Jlab

1          1     0      1
2          1     1      0
3          1     0      2
4          1    0.5    0 ;


variable COST;
positive variables Y;

equations
obj
RESBAL;

Obj .. Cost =e= sum (i,b(i)* Y(i));
RESBAL(j).. sum(i,a(j,i)* Y(i)) =G= c(j);

model farmdual/all/;
solve farmdual using lp minimizing cost;
```

```
               S O L V E      S U M M A R Y

     MODEL   farmdual               OBJECTIVE  COST
     TYPE    LP                     DIRECTION  MINIMIZE
     SOLVER  CPLEX                  FROM LINE  32

**** SOLVER STATUS     1 NORMAL COMPLETION
**** MODEL STATUS      1 OPTIMAL
**** OBJECTIVE VALUE             2640.0000

Optimal solution found.
Objective :          2640.000000

                        LOWER      LEVEL      UPPER     MARGINAL

---- EQU obj              .          .          .          1.000

---- EQU RESBAL

     LOWER      LEVEL      UPPER     MARGINAL
1    30.000     30.000     +INF      80.000
2    10.000     12.000     +INF        .
3    40.000     48.000     +INF        .
4    12.000     12.000     +INF      20.000

                        LOWER      LEVEL      UPPER     MARGINAL
---- VAR COST           -INF     2640.000     +INF        .

        LOWER      LEVEL      UPPER     MARGINAL

land      .        12.000     +INF        .
Mlab      .          .        +INF      90.000
Jlab      .        18.000     +INF
```