Duality

For ARE 521 QUANTITATIVE TECHNIQUES

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Plan

- Overview of the homework
  - Graphical solution
  - GAMS output

- Duality
  - Theory
  - Empirical example
  - Setting up dual problem in GAMS (time permitting)

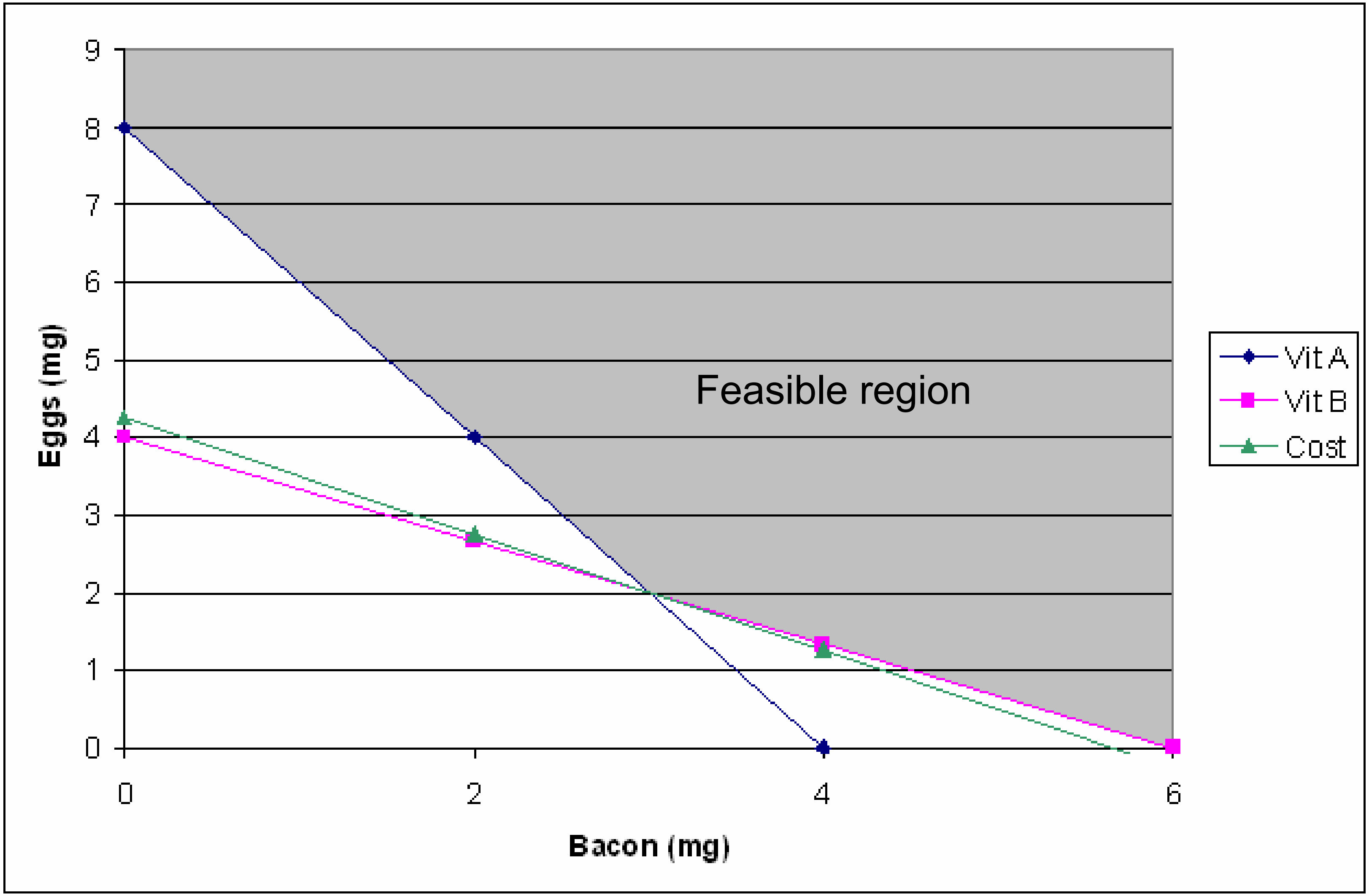
Graphical Solution

1. Prepare graph of feasible solutions for each of the constraint
2. Determine the feasible region by identifying the solutions that satisfy all the constraints simultaneously
3. Draw an objective function line showing the values of the decision variables that yield a specified value of the objective function
4. Move parallel objective function lines toward smaller objective function values until further movement would take the line completely outside the feasible region
5. Identify optimal level of decision variables and objective function

Based on Anderson et. al. *An Introduction to Management Science*

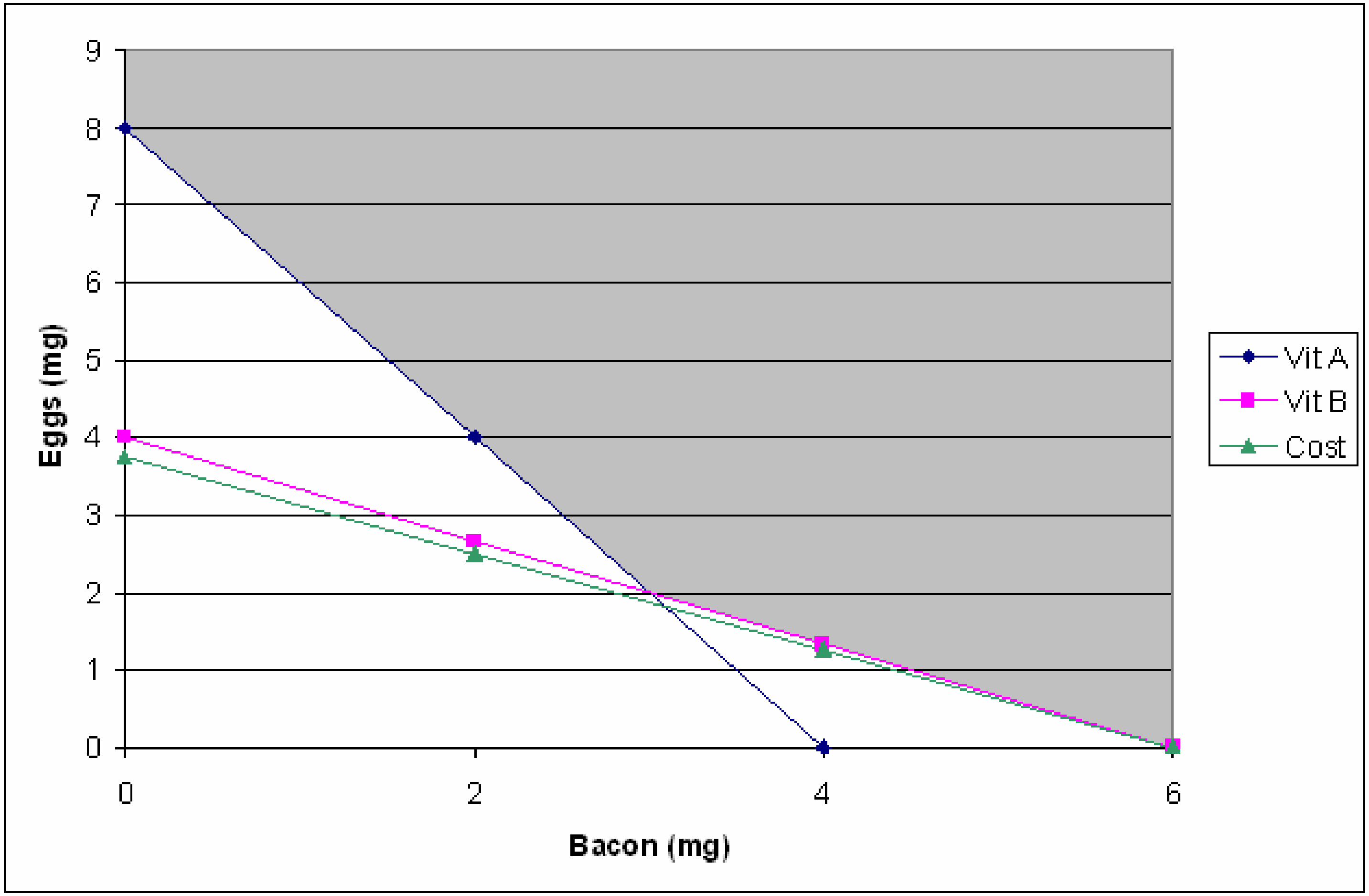
Graphical Solution

Objective function: *Cost = 4*Eggs + 3*bacon*
Optimal: 2 mg of eggs, 3 mg of bacon



Graphical Solution: Modified Problem

Objective function: *Cost = 4*Eggs + 2.5*Bacon*
Optimal: 0 mg of eggs, 6 mg of bacon



Graphical Solution (*cont.*)

- Original solution (2 mg of eggs, 3 mg of bacon) remain optimal as long as

slope of Vit A constraint ≤ slope of the objective function line ≤ slope of Vit B constraint

- Slope of the objective function line is usually negative; hence rotating the objective function line clockwise makes the line steeper even though the slope is getting smaller (more negative)

Recall: Sensitivity Analysis from Excel

Microsoft Excel 10.0 Sensitivity Report
Worksheet: [Assignment3Problem2.xls]Sheet1
Report Created: 9/26/2005 7:13:19 PM

Range of optimality

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B7	decision variables Eggs (mg)	2	0	4	0.5	2.5
C7	decision variables Bacon (mg)	3	0	3	5	0.33

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F3	A Actual Level	16	0.125	16	8	8
F4	B Actual Level	12	1.25	12	12	4

Range of Optimality for Objective Function Coefficients

- Provides the range of values over which the current solution will remain optimal

- Managerial attention: objective function coefficients with narrow range of optimality, and the coefficients near the end point of range. A small change can necessitate modifying optimal solution

LP Minimization problem

- Any excess quantity corresponding to ≥ constraint is referred to as *surplus*
- *Negative dual price* for constraint tells us that the objective function *will not improve* if the value of the right-hand side is increased by one unit
  - If the right-hand side of constraint increase by one unit, the value of objective function will get worse
  - Becoming worse in a minimization problem means increase in total cost
  - Caution: negative dual price can be converted to positive by multiplying the constraint by (-1)

Test:

- What is feasible region for equality constraint?

Things to look for in the LP output:

- Optimal value of *objective function*
- Optimal levels of *decision variables*

  ≤
- *Marginal values* of the constraints
- *Reduced cost* for non-basic variables
- Any slack / surplus in constraints

Marginal Values for the Constraints

- Assignment 3, problem 1
  - Find optimal football and basketball production to maximize Gross Margin given resource constraints:

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient
B2	decision variable value footballs	1.00	0.00	8.00
C2	decision variable value basketballs	3.33	0.00	10.00

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side
E4	Capital use Level at the optimum	15.00	0.44	15.00
E5	Labor use Level at the optimum	24.00	1.44	24.00

What shadow prices mean??

Marginal Values for the Constraints

- Assignment 3, problem 1
  - Find optimal football and basketball production to maximize Gross Margin given resource constraints:

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Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side
E4	Capital use Level at the optimum	15.00	0.44	15.00
E5	Labor use Level at the optimum	24.00	1.44	24.00

Shadow prices:

1. Increase in GM for every unit increase in resources;
2. Maximum price the producer can pay for extra unit of resource

Marginal Values for the Constraints (*cont.*)

- Assignment 3, problem 2:
  - Find optimal consumption of eggs and bacon to minimize breakfast cost while meeting minimum vitamin requirements

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient
B7	decision variables Eggs (mg)	2	0	4
C7	decision variables Bacon (mg)	3	0	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side
F3	A Actual Level	16	0.125	16
F4	B Actual Level	12	1.25	12

What shadow prices mean???

Marginal Values for the Constraints (*cont.*)

- Assignment 3, problem 2:
  - Find optimal consumption of eggs and bacon to minimize breakfast cost while meeting minimum vitamin requirements

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient
B7	decision variables Eggs (mg)	2	0	4
C7	decision variables Bacon (mg)	3	0	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side
F3	A Actual Level	16	0.125	16
F4	B Actual Level	12	1.25	12

Shadow prices:

1. Increase in breakfast cost for every unit increase in vitamin requirements
2. Minimum “satisfaction” associated with consumption of extra unit of vitamins

Final value of vitamin consumption = minimum requirements => no surplus

GAMS Output

Reports results, facilitates debugging

- Echo Print
- Reference Map
- Equation Listing
- Column Listing
- Model Statistics
- Status Report, Solver report
- Solution Report

Echo Print

- Copy of input file with line numbers
- Can be blocked by a *dollar-print-control statement*

$Offlisting

- Dollar-print-control statements
  - output control
  - start in the very beginning of a line
  - Other examples:

$Title   TEXT   print TEXT on top of each page

$Ontext   comments

$Offtext

Reference Map

- Alphabetical, cross-referenced list of all the entities (sets, parameters, variables, and equations) of the model.
- Example:

SYMBOL	TYPE	REFERENCES					
A	PARAM	DECLARED	9	DEFINED	10	REF	42
B	PARAM	DECLARED	13	DEFINED	14	REF	44

- symbol A is a parameter that was declared in line 9, defined (assigned value) in line 10, and referenced in line 42.

Equation Listing

- Describe equations for specific values of set elements and parameters
- Does GAMS generate the model you intended?
- Nonlinear system – first order (linear) Taylor approximation
- Example:

```
---- const  =G=  constraints

const(A)..  2*x(eggs) + 4*x(bacon) =G= 16 ; (LHS = 0, INFES = 16 ***)
```

Column Listing

- shows the coefficients of three specific variables for each generic variable

- Example

```
---- x  products

x(footballs)
            (.LO, .L, .UP = 0, 0, +INF)
     -8       obj
      5       const(capital)
      4       const(labor)
```

- ***control of equation and column listing:***

  option limrow = r, limcol = c ;

  r – desired number of equations

  c – desired number of columns

Note: *Variable attributes*

Variable attribute	Variable suffix	Description
lower bound	`.lo`	The lower bound for the variable. Set by the user either explicitly or through default values.
upper bound	`.up`	The upper bound for the variable. Set by the user either explicitly or through default values.
fixed value	`.fx`	The fixed value for the variable.
activity level	`.l`	The activity level for the variable. This is also equivalent to the current value of the variable. Receives new values when a model is solved.
marginal or dual value	`.m`	The marginal value for the variable. Receives new values when a model is solved.
scale value	`.scale`	This is the scaling factor on the variable. This is normally an issue with nonlinear programming problems and is discussed in detail later.
branching priority value	`.prior`	This is the branching priority value of a variable. This parameter is used in mixed integer programming models only, and is discussed in detail later.

Examples: **display** x.L; x.fx = 1;

Low and Upper bounds for variables

Variable Type	Allowed Range of Variable
free(default)	$-\infty$ to $+\infty$
positive	0 to $+\infty$
negative	$-\infty$ to 0
binary	0 or 1
integer	0,1,...., 100 (default)

Model Statistics

- BLOCK counts
  - number of generic equations and variables
- SINGLE counts
  - refer to individual equations and variables
- NON ZERO ELEMENTS
  - number of non-zero coefficients in the problem matrix
- GENERATION TIME
  - time used since the syntax check finished

Solver Status

solvestat	solver status
1	normal completion → desired
2	iteration interrupt
3	resource interrupt
4	terminated by solver
5	evaluation error limit
6	unknow
7	(unused)
8	error preprocessor error
9	error setup failure
10	error solver failure
11	error internal solver error
12	error post-processor error
13	error system failure

Model Status

modelstat	model status	
1	optimal	→ Desired for LP
2	locally optimal	→ Desired for NLP
3	unbounded	
4	infeasible	
5	locally infeasible	
6	intermediate infeasible	
7	intermediate non-optimal	
8	integer solution	→ Desired for MIP
9	intermediate non-integer	
10	integer infeasible	
11	(unused)	
12	error unknown	
13	error no solution	

Solver Report

- message identifying the solver and its authors
- diagnostic messages if anything unusual was detected
- specific performance details
- more about various GAMS solvers is at http://www.gams.com/solvers/solvers.htm

Solution Report

- Results of optimization
- Four levels of equations – low bound, level value, upper bound, and marginal
- Values

  "." – zero

  EPS – close to zero

  INFES - infeasible

  NOPT - marginal values of the wrong sign

  UNBND – unbounded

- Turned off by line:

  option solprint = off

Note: *Equation Values*

Type	.lo	.up
`=e=`	rhs	rhs
`=l=`	-inf	rhs
`=g=`	rhs	inf
`=n=`	-inf	inf

- Example

---- EQU const resource constraints

	LOWER	LEVEL	UPPER	MARGINAL
capital	-INF	15.000	15.000	0.444

Help to identify slack or surplus in the constraints!

Report Summary

- total number of non-optimal, infeasible, and unbounded rows and columns
- INFES – row/column is infeasible. The level value is not between upper and lower bounds
- NOPT – row/column is non-optimal. Marginal value is incorrect
- UNBND – row/column is unbounded

Debugging Tips

- Always check carefully for the cause of the first error
- Look at the previous line (especially for missing semicolons) if nothing seems obvious
- More at:

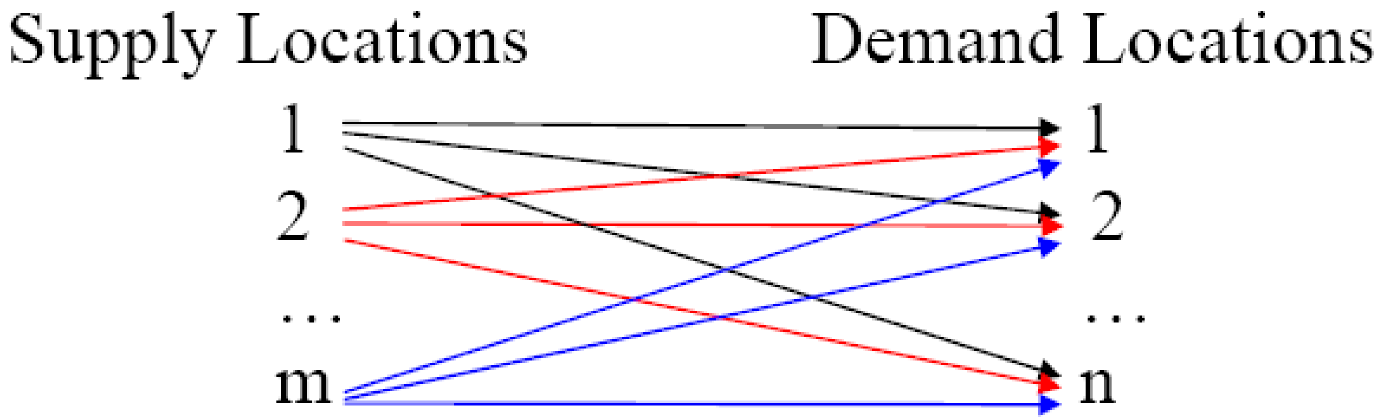
http://agecon2.tamu.edu/people/faculty/mccarl-

bruce/641clas/04_641_model_inspection_error.pdf

Duality

Transportation Problem: Primal

- This problem involves the shipment of a homogeneous product from a number of supply locations to a number of demand locations.



- Problem: given needs at the demand locations, how should we take limited supply at supply locations and move the goods. Further suppose we wish to minimize cost.

Based on McCarl 2005

General LP Form

$$Min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i ,$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j ,$$

$$x_{ij} \geq 0$$

where

i – supply locations, j – demand locations,

c_{ij} – unit shipping cost, x_{ij} – shipment volumes,

b_j – demand levels, a_i – supply limits

Transportation Problem: *Dual*

- Suppose that the shipping contract calls for the trucking company to actually buy the product at the origins and sell it at the destinations at whatever price it is able to pay and receive, given that the transportation costs on all routes are fixed and known. What prices p_j^d and p_i^s should the manager be prepared to take and offer at the destinations and origins, respectively?

The Dual of the Transportation Problem: *Empirical Example (cont)*

- Prices that maximize the difference between
  - the value of goods sold at the markets, and
  - the value of the same goods bought from producers

- Maximize the value added of the fish distributor

$$\text{maximize}\begin{bmatrix}\text{value of goods} & & \text{value of goods} \\ \text{transported and sold} & - & \text{bought} \\ \text{at the markets} & & \text{from producers}\end{bmatrix}$$

- The company will not transport product if the price difference between origin and destination is less than unit transportation cost
- If the price difference is greater than the transportation cost, it is not a competitive equilibrium

Basic Concept

- Objective:
  - » Maximize Value Added
- Variables:
  - » Prices at each supply and demand points
- Restrictions:
  - » Non negative prices
  - » Links between demand and supply prices and transportation costs

Based on Paris 1995

Formulating the Dual Problem: *the decision variables*

- Let us define our decision variable as the two sets of prices: prices at supply location *i* (p^s_i) and prices at demand location *j* (p^d_j)

Formulating the Dual Problem: *The objective function*

- We want to maximize value added
- Which is the *sum* of values of products values sold at all destinations *minus* the *sum* of values of product bough at all origins

Maximize $\sum_{demand_j} p^d_j \, b_j - \sum_{supply_i} p^s_i \, a_i$

Where *b_i* denotes demand at markets, and *a_i* stands for supply limits

Formulating the Dual Problem: *Constraints*

- Links between demand and supply prices and transportation costs – equilibrium point

$$p^d_j - p^s_i \leq c_{ij}$$

for all *i*'s and *j*'s, where c_{ij} refers to unit transportation cost

- nonnegative prices

p^d_j and $p^s_i \geq 0$ for all *i*'s and *j*'s

$$\text{Maximize} \sum_{demand_j} p^d_j \, b_j - \sum_{supply_i} p^s_i \, a_i$$

$p^d_j - p^s_i \leq c_{ij}$ for all *i*'s and *j*'s

p^d_j and $p^s_i \geq 0$ for all *i*'s and *j*'s

Primal and Dual

$$Min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$-\sum_{j=1}^{n} x_{ij} \geq -a_i,$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j,$$

$$x_{ij} \geq 0$$

$$Max \sum_{j=1}^{n} b_j p_j^d - \sum_{i=1}^{m} a_i p_i^s$$

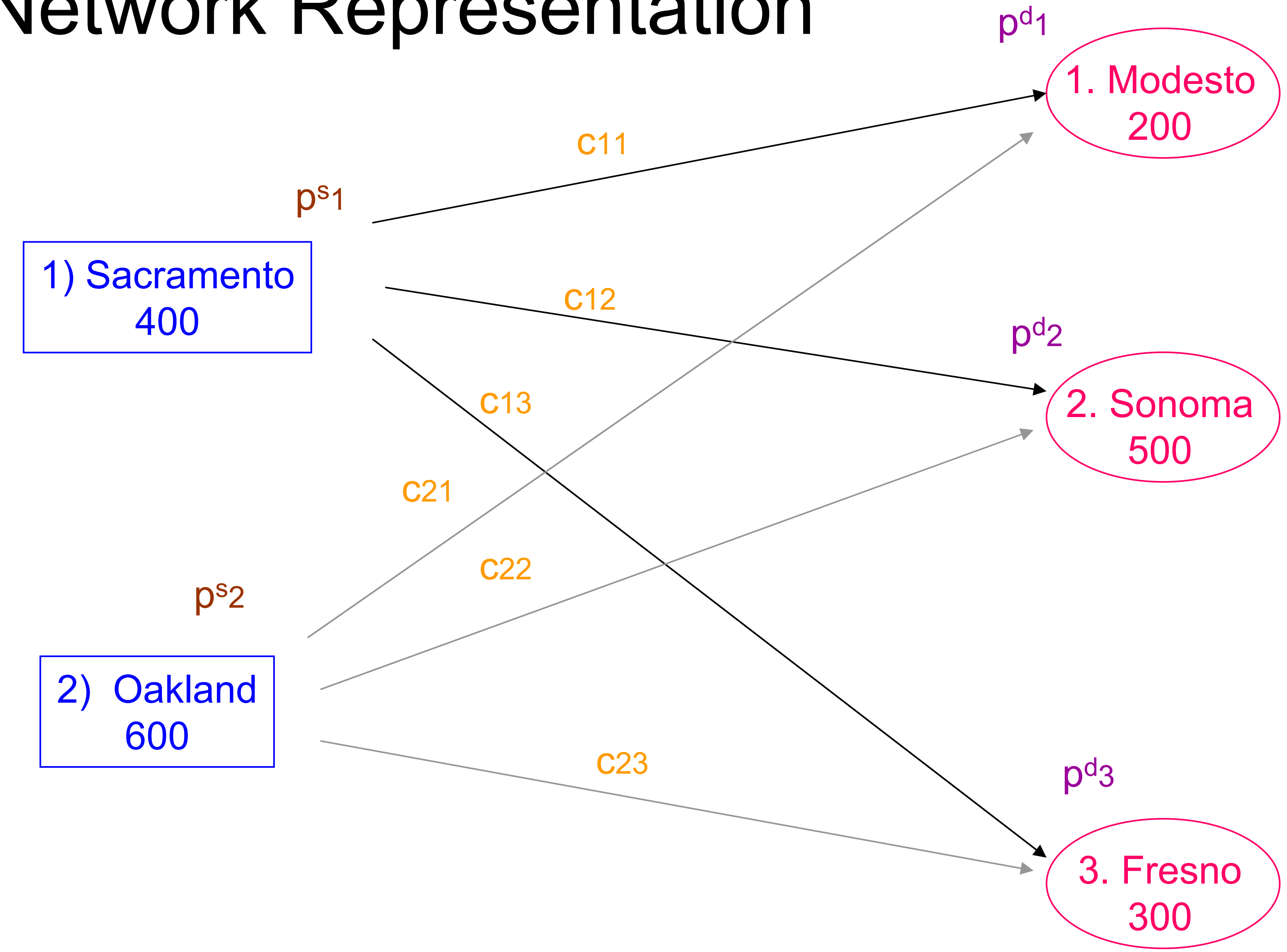
$$p_j^d - p_i^s \leq c_{ij},$$

$$p_j^d, p_i^s \geq 0$$

Primal and Dual of Transportation Problem (cont.)

	Primal	*Dual*
Objective	Min	Max
Equation type	≥	≤
Demand and supply limits	RHS of constraints	Coefficient in objective function
Decision variables	Shipment quantities	Prices

Network Representation



Based on Paris 1991

Example:
Problem Formulation

- Objective function:
  - maximize value added

Max

$p^d_1 b_3 + p^d_2 b_2 + p^d_3 b_3 - p^s_1 a_1 - p^s_2 a_2$

$$MaxVA = \sum_j p^d_j b_j{}^d - \sum_i p^s_i a_i$$

- Equilibrium constraints

$p^d_1 - p^s_1 \leq c_{11}$

$p^d_1 - p^s_2 \leq c_{12}$

$p^d_2 - p^s_1 \leq c_{21}$

$p^d_2 - p^s_2 \leq c_{22}$

$p^d_3 - p^s_1 \leq c_{13}$

$p^d_3 - p^s_2 \leq c_{23}$

$$p^d_j - p^s_i \leq c_{ij}$$

for all *j* and *i*

Example: Tableau

$p^d{}_1$	$p^d{}_2$	$p^d{}_3$	$p^s{}_1$	$p^s{}_2$		
200	500	300	- 400	- 600		max
1			-1		≤	40
1				-1	≤	40
	1		-1		≤	50
	1			-1	≤	30
		1	-1		≤	80
		1		-1	≤	90

Example: *Optimal Pricing*

Destinations						Origins			
Modesto		Sonoma		Fresno		Sacramento		Oakland	
unit	variable	unit	variable	unit	variable	unit	variable	unit	variable
40	p^d_1	**30**	p^d_2	**80**	p^d_3	**0**	p^s_1	**Close to zero**	p^s_2

Example: *Optimal Solution*

- Objective value $47,000

Variable	*Value*	*Reduced cost*
$Pdemand_1$	40	0
$Pdemand_2$	30	0
$Pdemand_3$	80	0
$Psupply_1$	0	0
$Psupply_2$	0	Close to zero

Equation	*Slack*	*Shadow price*
$(demand_1,supply_1)$	0	100
$(demand_1,supply_2)$	0	100
$(demand_2,supply_1)$	20	0
$(demand_2,supply_2)$	0	500
$(demand_3,supply_1)$	0	300
$(demand_1,supply_2)$	10	0

Example: *Solution*

- shadow price – change in objective function for 1 unit change in RHS of constraint
  - i.e. reduction in unit cost of transportation from Sacramento to Modesto would decrease value added by 100 (due to change in prices for other markets)
- reduced cost represents marginal costs of forcing nonbasic variable into the solution
  - i.e. increase in supply price in supply locations would not have any effect on value added

Example: *Shadow prices*

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B9	modesto, sacramento	40	100	40	0	10
B10	modesto, oaklan	40	100	40	10	0
B11	sonoma, sacramento	30	0	50	1E+30	20
B12	sonoma, oakland	30	500	30	20	30
B13	freson, sacramento	80	300	80	10	80
B14	fresno, oakland	80	0	90	1E+30	10

References

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