

ARE 521 QUANTITATIVE TECHNIQUES

Dynamic Programming

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10/9/2005

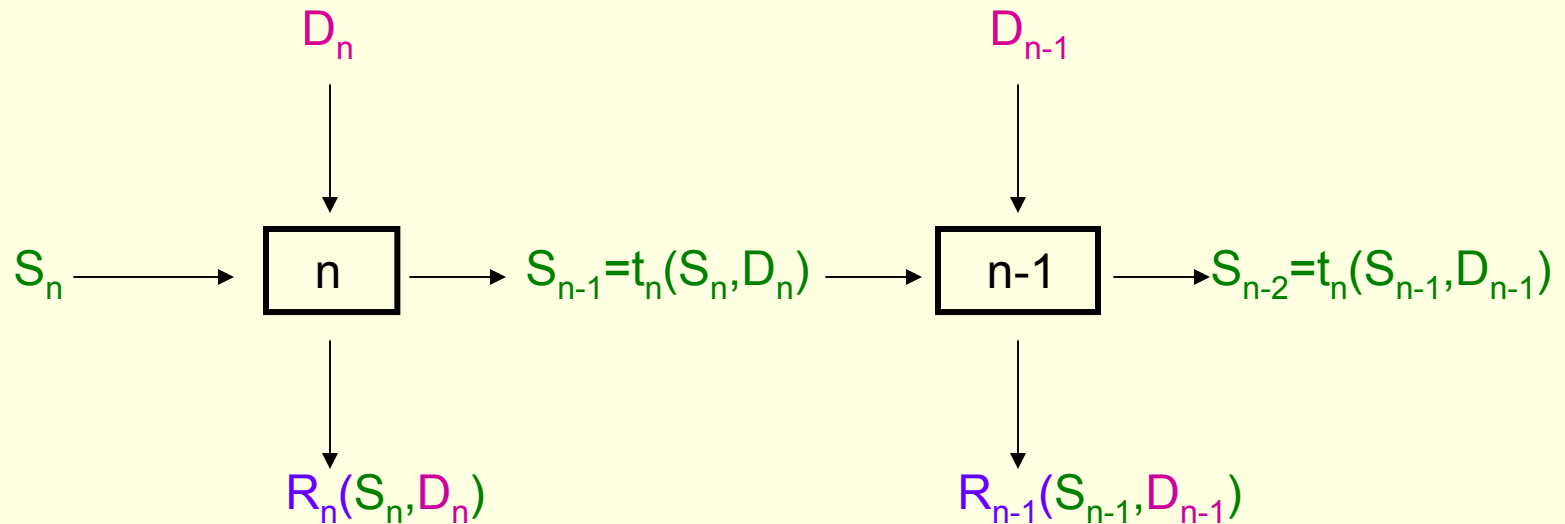
Dynamic Optimization

- Can be viewed as a multistage decision making process
- solve sequential, irreversible and risky decisions
- Optimal solution would involve more than one single value for choice variable
- Derive optimal time path for every choice variable

Approaches to Dynamic Optimization

- The Calculus of Variation
 - John and James Bernoulli
- Optimal control theory
 - Based on Pontryagin (Russian Mathematician) maximization principle
- Dynamic Programming
 - Based on Bellman Principle of Optimality

Components of a DP problem



D_n : Decision at stage n

S_n : State at stage n

R_n : Return at stage n

$S_{n-1} = t_n(S_n, D_n)$: Transition function

Dynamic Programming

- Bellman's Principle of Optimality

“ An optimal policy or a set of decisions has the property that whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision ”

- Separation between **stage return** and **the state transformation** at stage n from the decisions and states at other stages
- Complete set of decisions are handled in a sequential way rather than a simultaneous way
- Based on the assumption of separability or **Markovian assumption**

DP vs LP

■ advantages :

- not restricted to linear relationships (non linear stage return and stated transformation functions)
- efficient mean of solving problems with many stages
- Can be readily extended accommodate problems in stochastic nature

Limitations

- Assumption of separability : the returns and the state transformation at a stage depend only on the decision made in that stage and the state in that stage

Application of DP to Natural Resources

Examples of Functions in Some Resource Control Problems

<i>Natural resource</i>	x_i	u_i	$g_i\{x_i\}$	$h_i\{x_i, u_i\}$	$a_i\{x_i, u_i\}^a$
Mine	Resource mass	Level of extraction	—	Level of extraction	Net returns from extraction = $p_i h_i - C_i\{x_i, u_i\}$
Irrigated crop	Crop biomass	Level of irrigation	—	Irrigation induced growth ($h_i < 0$)	Net returns from irrigation = $-p_n h_i (1+r)^{n-i} - c_i u_i$
Beef cattle	Liveweight	Level of feed	—	Weight gain ($h_i < 0$)	Net returns from feeding = $-p_n h_i (1+r)^{n-i} - c_i u_i$
Timber thinning	Timber biomass	Level of thinning	Autonomous growth	Level of thinning	Net returns from thinning = $(p_i - c_i) u_i$
Fishery	Fish biomass	Level of fishing effort	Autonomous growth	Level of harvesting	Net returns from harvesting = $p_i h_i - C_i\{x_i, u_i\}$

^a p_i = price of product; c_i = cost per unit of control; $C_i\{\cdot\}$ = total control cost; r = rate of discount.

Representation of DP Problem

$$\underset{u_1 \dots u_n}{Max} \sum_{i=1}^n \alpha^{i-1} a_i \{x_i, u_i\} + \alpha^n F \{x_{n+1}\}$$

Subject to

$$x_1 = \overline{x_1}$$

$$x_{i+1} = t_i \{x_i, u_i\} \quad (i = 1, \dots, n)$$

Where

Sequence of decision to be made : $u_i \dots u_n$

Transformation function : $x_{i+1} = x_i + g_i \{x_i\} - h_i \{x_i, u_i\}$

Final value associated with the terminal state : $F \{x_{n+1}\}$

Autonomous addition to resource stock between stage i stage $i+1$: $g_i \{x_i\}$

Control reduction in the resource stock between stage i stage $i+1$: $h_i \{x_i, u_i\}$

Stage return $a_i \{x_i, u_i\}$

Corresponding Recursive Equation

$$v_i \{x_i\} = \underset{u_i}{Max} \left[\alpha^{i-1} a_i \{x_i, u_i\} + \alpha V_{i+1} \{t_i \{x_i, u_i\}\} \right]$$

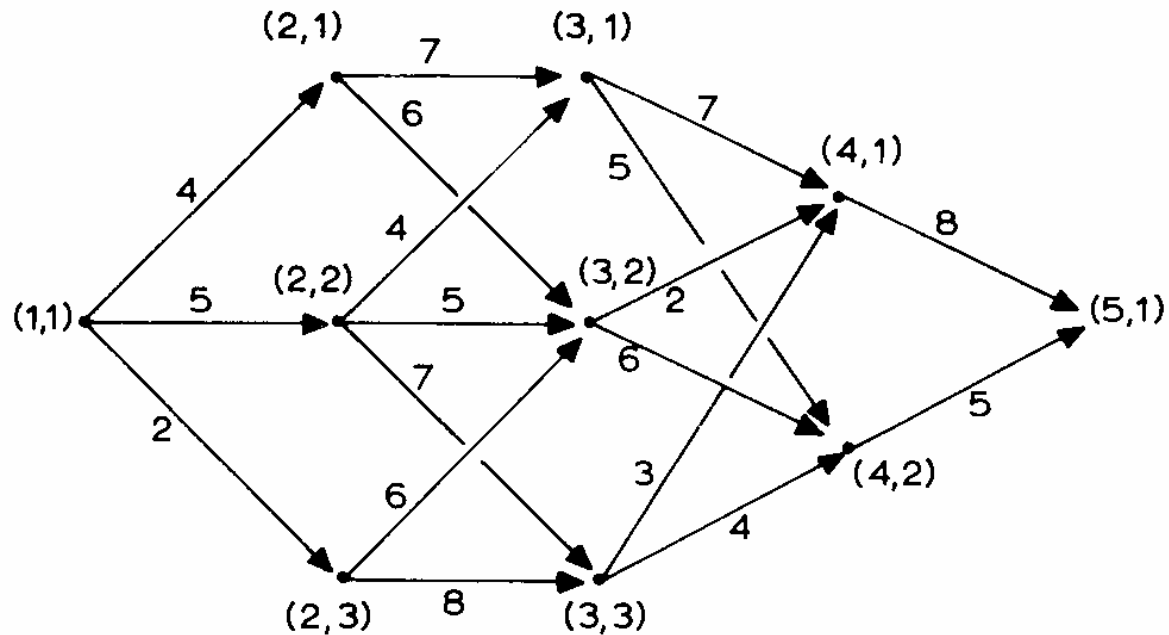
With

$$V_{n+1} \{x_{n+1}\} = F \{x_{n+1}\} \quad (i = n, \dots, 1)$$

$$x_1 = \bar{x}_1$$

Example : A Least cost network Problem

$$Z = \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij} + F(x)$$



A least-cost network problem.

$$\min_{J_2, J_3, J_4} \sum_{i=1}^4 a_i \{J_i, J_{i+1}\} + F\{J_4\}$$

where $a_i \{J_i, J_{i+1}\}$ is the cost of linking (i, J_i) and $(i+1, J_{i+1})$; and

$$F\{J_4\} = \begin{cases} 8 & \text{if } J_4 = 1 \\ 5 & \text{if } J_4 = 2 \end{cases}$$

The recursive equation is

with
$$V_i \{J_i\} = \min_{J_{i+1}} [a_i \{J_i, J_{i+1}\} + V_{i+1} \{J_{i+1}\}] \quad (i = 3 \text{ to } 1)$$

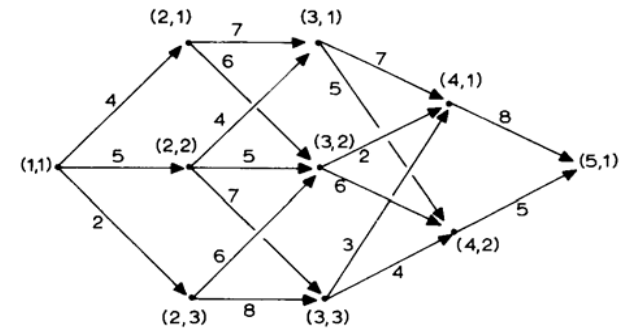
$$V_4 \{J_4\} = F\{J_4\}$$

$$V_1 \{J_1\} = \min_{J_2} \{a_1 \{J_1, J_2\} + V_2 \{J_2\}\}$$

$$V_2 \{J_2\} = \min_{J_3} \{a_2 \{J_2, J_3\} + V_3 \{J_3\}\}$$

$$V_3 \{J_3\} = \min_{J_4} \{a_3 \{J_3, J_4\} + V_4 \{J_4\}\}$$

Optimal path



A least-cost network problem.

Cost to (5,1) from Stage -3 Nodes

Node	Next Node	Least cost	Optimal next mode
	(4,1)	(4,2)	
		V3{J3}	
(3,1)	→ (7+8)	15	(5+5)
(3,2)	(2+8)	10*	(6+5)
(3,3)	(3+8)	11	(4+5)

: Linking Nodes (3,1) to (4,1), $a_3 \{1,1\}, 7 + V_4 \{1\}, 8 = 15$

: Linking Nodes (3,1) to (4,2), $a_3 \{1,2\}, 5 + V_4 \{2\}, 5 = 10$

Optimal Path

Cost to (5,1) from Stage -2 Nodes

Node	Next Node						Least cost	Optimal next mode
	(3,1)		(3,2)		(3,3)			
							V2{J2}	
(2,1)	(7+10)	17	(6+10)	16*			16	(3,2)
(2,2)	(4+10)	14*	(5+10)	15	(7+9)	16	14	(3,1)
(2,3)			(6+10)	16*	(8+9)	17	16	(3,2)

Cost to (5,1) from Stage -1 Nodes

Node	Next Node						Least cost	Optimal next mode
	(2,1)		(2,2)		(2,3)			
							V1{J1}	
(1,1)	(4+16)	20	(5+14)	19	(2+16)	18*	18	(2,3)

Optimal Path

■ $(1,1) \longrightarrow (2,3) \longrightarrow (3,2) \longrightarrow (4,1) \longrightarrow (5,1)$

Minimum cost = $2 + 6 + 2 + 8 = 18$

References

- Kennedy J.O.S. (1986)- *Dynamic programming , Application to Agriculture and Natural Resources*
- <http://www.business.latrobe.edu.au/public/staffhp/jkennedy/>
- Chiang A. C. (1992) - Elements of Dynamic Optimization

Optimal Irrigation : Kennedy (1986)

A farmer grows three horticultural crops in successive seasons over one year on 100 ha. Each crop takes four months, or one season, to reach maturity from the time of planting. The yield of each crop (in hundreds of tonnes per 100 ha) is given by

$$y_i = w_i - 0.1w_i^2 \quad (i = 1 \text{ to } 3)$$

where w_i is the depth of water in centimetres received by the crop grown in the i -th season. The depth of water received depends on the height of

water released from storage at the beginning of each season (u_i in metres) and rainfall received during each season (q_i in centimetres). The area of the dam is 1 ha, so

$$w_i = u_i + q_i$$

The dam is full at the beginning of the first season with a water height of 3 m. The amount of water which can be released at the beginning of any season is limited to integer values of metres of water, and by the amount in storage. Rainfall augments the water in storage. The catchment area is 100 ha, so 1 cm of rainfall raises the level of the dam (x_i) by 1 m, provided the dam is not full.

The farmer's objective is to select integer values of u_i so that the present value of receipts from sale of the crops is maximized. The price (in dollars per tonne) received for the i -th season crop is b_i . The problem can be formulated as:

$$\max_{u_1, u_2, u_3} \sum_{i=1}^3 \alpha^{i-1} b_i (w_i - 0.1 w_i^2)$$

subject to

$$0 \leq u_i \leq x_i \leq 3, \quad u_i, x_i \text{ integer}$$

$$x_1 = 3$$

$$x_{i+1} = \min(x_i - u_i + q_i, 3)$$

with data

$$b = [50 \quad 100 \quad 150]$$

$$q = [2 \quad 1 \quad 1]$$

$$\alpha = 0.95$$

The backward recursive equation used to solve this problem is

$$V_i\{x_i\} = \max_{0 \leq u_i \leq x_i} [b_i(w_i - 0.1 w_i^2) + \alpha V_{i+1}\{x_i - u_i + q_i\}] \quad (i = 3 \text{ to } 1) \quad (2.3)$$

with

$$w_i = u_i + q_i$$

$$V_4\{x_4\} = 0$$

Irrigation Problem

Optimal irrigation returns (\$000) _ Satge3

Dam water Level	Water released (ha ,m)												V3(X3)	U*{3}
	U3	0		U3	1		U3	2		U3	3			
	q3	w3		q3	w3		q3	w3		q4	w3			
	1	1		1	2		1	3		1	4			
X3	Rev	V{X4}		Rev	V{X4}		Rev	V{X4}		Rev	V{X4}			
0	13.5	0.0	13.5										13.5	0
1	13.5	0.0	13.5	24.0	0.0	24.0							24.0	1
2	13.5	0.0	13.5	24.0	0.0	24.0	31.5	0.0	31.5				31.5	2
3	13.5	0.0	13.5	24.0	0.0	24.0	31.5	0.0	31.5	36.0	0.0	36.0	36.0	3

Optimal Irrigation Returns (\$000)- Stage 2

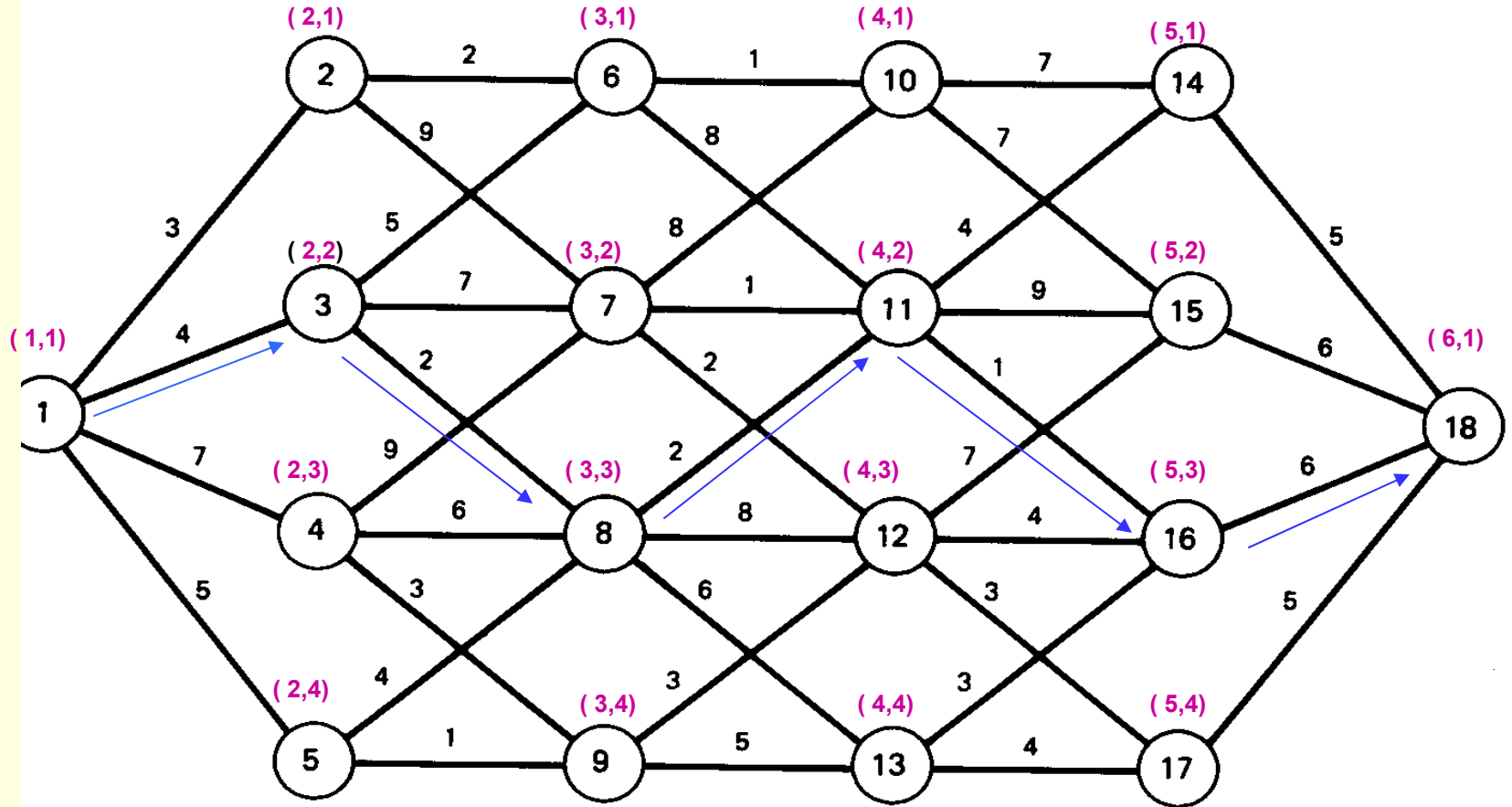
Dam water Level	Water released (ha ,m)													V2(X2)	U*{2}
	U2	0		U2	1		U2	2		U3	3				
	q3	w3		q3	w3		q3	w3		q4	w4				
	1	1		1	2		1	3		1	4				
X2	Rev	V{X3}		Rev	V{X3}		Rev	V{X3}		Rev	V{X3}				
0	9.0	22.8	31.8											31.8	0
1	9.0	29.9	38.9	16.0	22.8	38.8								38.9	0
2	9.0	34.2	43.2	16.0	29.9	45.9	21.0	22.8	43.8					45.9	1
3	9.0	34.2	43.2	16.0	34.2	50.2	21.0	29.9	50.9	24.0	22.8	46.8	50.9	50.9	2

Optimal Irrigation Returns (\$000)- Stage 1

Dam water Level	Water released (ha ,m)													V1(X1)	U*{1}
	U1	0		U1	1		U1	2		U1	3				
	q3	w3		q3	w3		q3	w3		q4	w4				
	2	2		2	3		2	4		2	5				
X1	Rev	V{X2}		Rev	V{X2}		Rev	V{X2}		Rev	V{X2}				
0	8.0	43.6	51.6											51.6	0
1	8.0	48.4	56.4	10.5	43.6	54.1								56.4	0
2	8.0	48.4	56.4	10.5	48.4	58.9	12.0	43.6	55.6					58.9	1
3	8.0	48.4	56.4	10.5	48.4	58.9	12.0	48.4	60.4	12.5	43.6	56.1		60.4	2

Season	Dam water level	water released	Season Return	Discounted Total return
	m	ha/m		\$0
1	3	2	12.0	60.4
2	3	2	21.0	50.9
3	2	2	31.5	31.5

Homework



$$Min_{J_2, J_3, J_4, J_5,} \sum_{i=1}^5 a_i \{ J_i J_{i+1} \} + F \{ J_5 \}$$

$$F \{ J_5 \} = 5 \text{ if } [J_5] = 1$$

$$= 6 \text{ if } [J_5] = 2$$

$$= 6 \text{ if } [J_5] = 3$$

$$= 5 \text{ if } [J_5] = 4$$

Optimal distance to (6,1) form stage 4_Nodes														
Node	Next Node												LC	O. N. M
	(5,1)			(5,2)			(5,3)			(5,4)				
(4,1)	7	5	12	7	6	13							12	(5,1)
(4,2)	4	5	9	9	6	15	1	6	7				7	(5,3)
(4,3)				7	6	13	4	6	10	3	5	8	8	(5,4)
(4,4)							3	6	9	4	5	9	9	(5,3), (5,4)

Optimal path

Optimal distance to (6,1) form stage 3_Nodes														
Node	Next Node												LC	O. N. M
	(4,1)			(4,2)			(4,3)			(4,4)				
(3,1)	1	12	13	8	7	15							13	(4,1)
(3,2)	8	12	20	1	7	8	2	8	10				8	(4,2)
(3,3)				2	7	9	8	8	16	6	9	15	9	(4,2)
(3,4)							3	8	11	5	9	14	11	(4,3)

Optimal distance to (6,1) form stage 2_Nodes														
Node	Next Node												LC	O. N. M
	(3,1)			(3,2)			(3,3)			(3,4)				
(2,1)	2	13	15	9	8	17							15	(3,1)
(2,2)	5	13	18	7	8	15	2	9	11				11	(3,3)
(2,3)				9	8	17	2	9	11	4	11	15	11	(3,3)
(2,4)							3	8	11	5	9	14	11	(3,3)

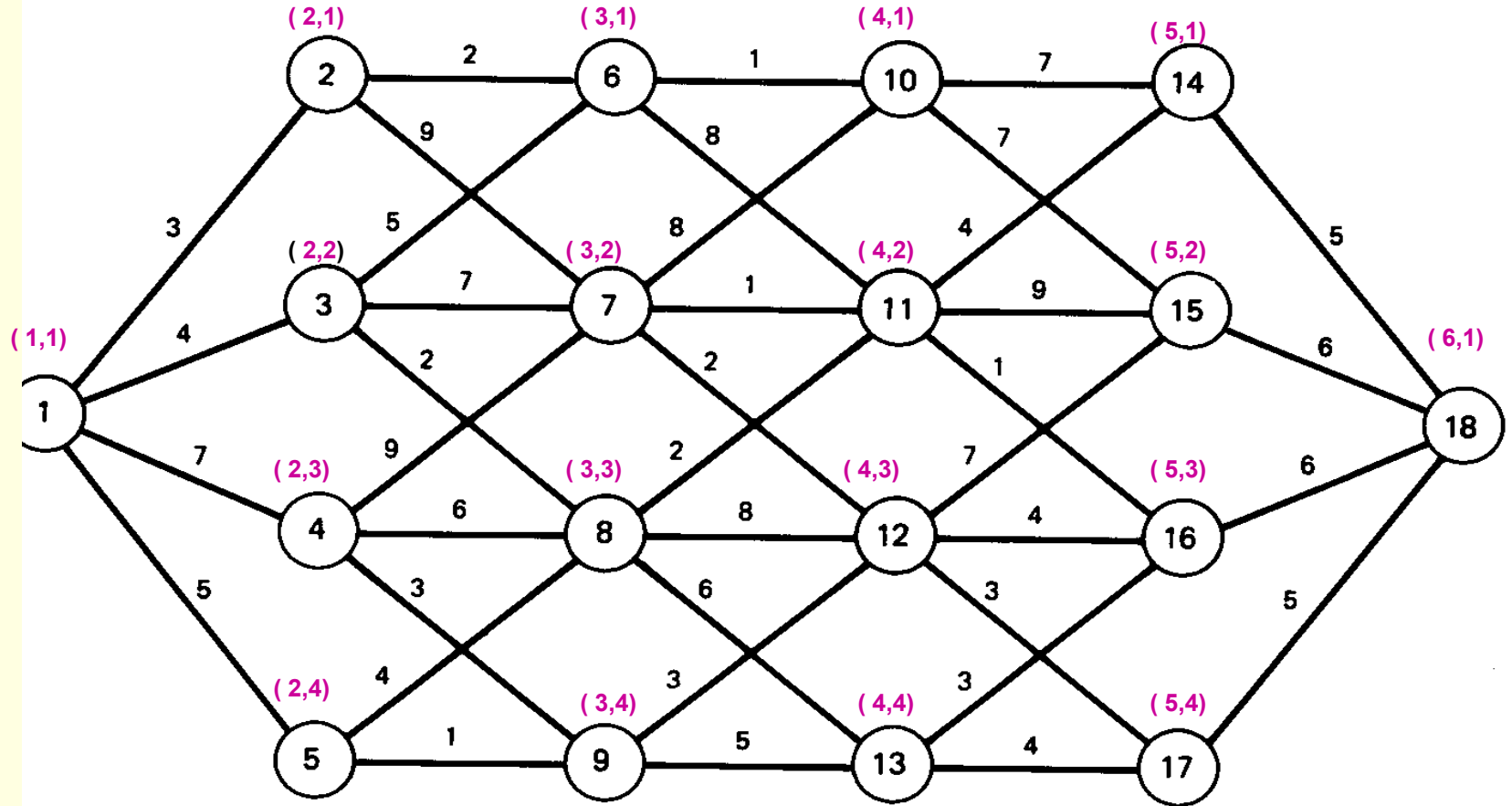
Optimal Path

Optimal distance to (6,1) form stage 1_Nodes														
Node	Next Node												LC	O. N. M
	(2,1)			(2,2)			(2,3)			(2,4)				
(1,1)	3	15	18	4	11	15	7	11	18	5	11	16	15	(2,2)

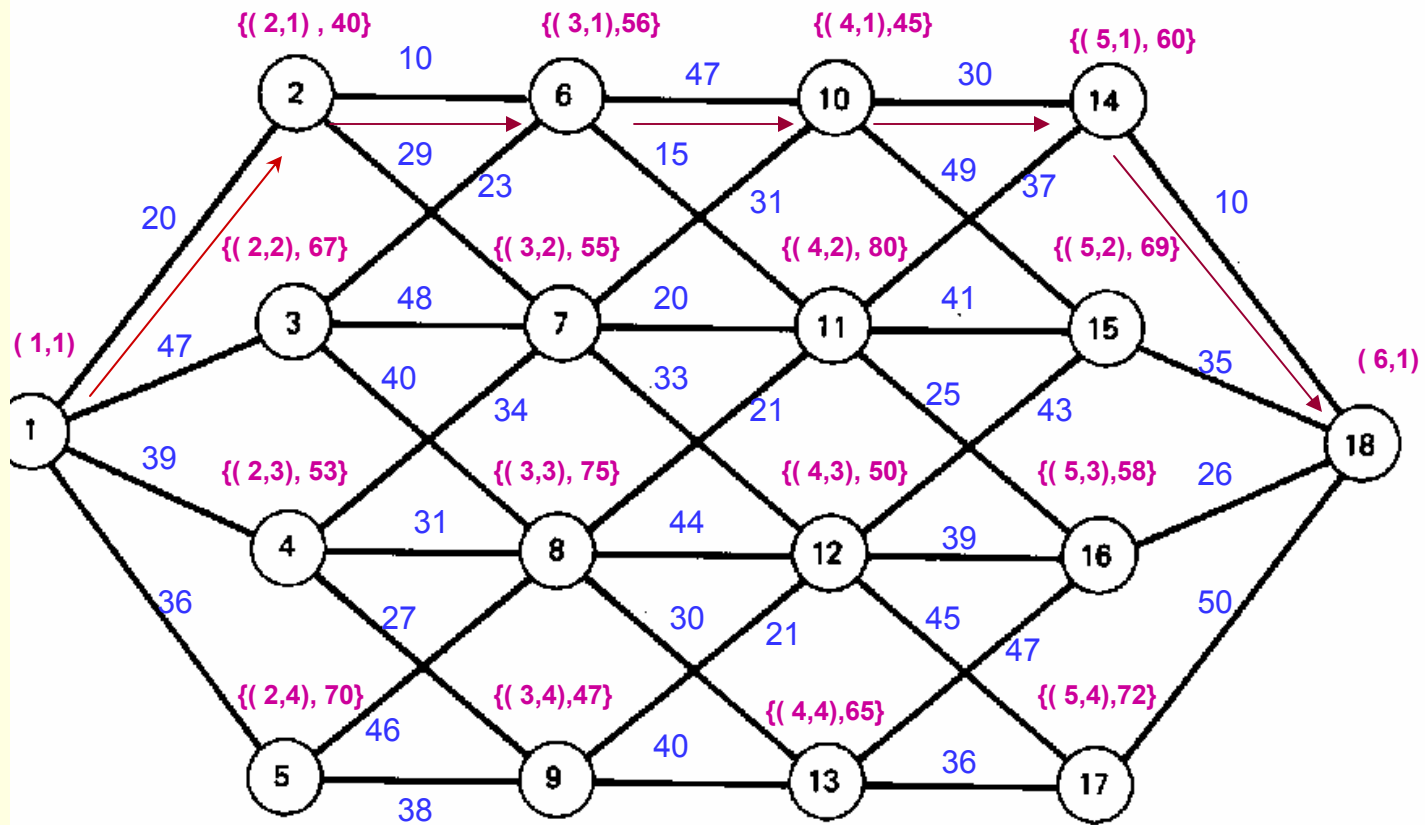
1 → 3 → 8 → 11 → 16 → 18

Shortest distance 1500 miles

Homework



Minimum cost



Minimum Cost

Cost to (6,1) form stage 4_Nodes														
Node	Next Node												LC	O. N. M
	(5,1)			(5,2)			(5,3)			(5,4)				
(4,1)	75	70	145	94	104	198							145	(5,1)
(4,2)	129	70	199	121	104	225	105	84	189				189	(5,3)
(4,3)				93	104	197	89	84	173	95	122	217	173	(5,3)
(4,4)							112	84	196	101	122	223	196	(5,3)

Cost to (6,1) form stage 3_Nodes														
Node	Next Node												LC	O. N. M
	(4,1)			(4,2)			(4,3)			(4,4)				
(3,1)	103	145	248	71	189	260							248	(4,1)
(3,2)	86	145	231	75	189	264	88	173	261				231	(4,1)
(3,3)				96	189	285	119	173	292	105	196	301	285	(4,2)
(3,4)							77	173	250	87	196	283	250	(4,3)

Cost to (6,1) form stage 2_Nodes														
Node	Next Node												LC	O. N. M
	(3,1)			(3,2)			(3,3)			(3,4)				
(2,1)	50	248	298	69	231	300							298	(3,1)
(2,2)	96	248	344	115	215	330	107	285	392				330	(3,2)
(2,3)				87	215	302	84	285	369	80	250	330	302	(3,2)
(2,4)							116	285	401	108	250	358	358	(3,4)

Cost to (6,1) form stage 1_Nodes															
Node	Next Node													LC	O. N. M
	(2,1)			(2,2)			(2,3)			(2,4)					
(1,1)	20	298	318	47	330	377	39	302	341	36	358	394	318	(2,1)	

Optimal Solution

Station to be constructed	Cons Cost	L Cost	Total cost
1		20	20
2	40	10	50
7	56	47	103
10	45	30	75
14	60	10	70
	201	117	318

Minimum total cost : 318, 000