ARE 521 QUANTITATIVE TECHNIQUES

Dynamic Programming

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Dynamic Optimization

- Can be viewed as a multistage decision making process
- solve sequential, irreversible and risky decisions
- Optimal solution would involve more than one single value for choice variable
- Derive optimal time path for every choice variable

Approaches to Dynamic Optimization

- The Calculus of Variation

  \- John and James Bernoulli

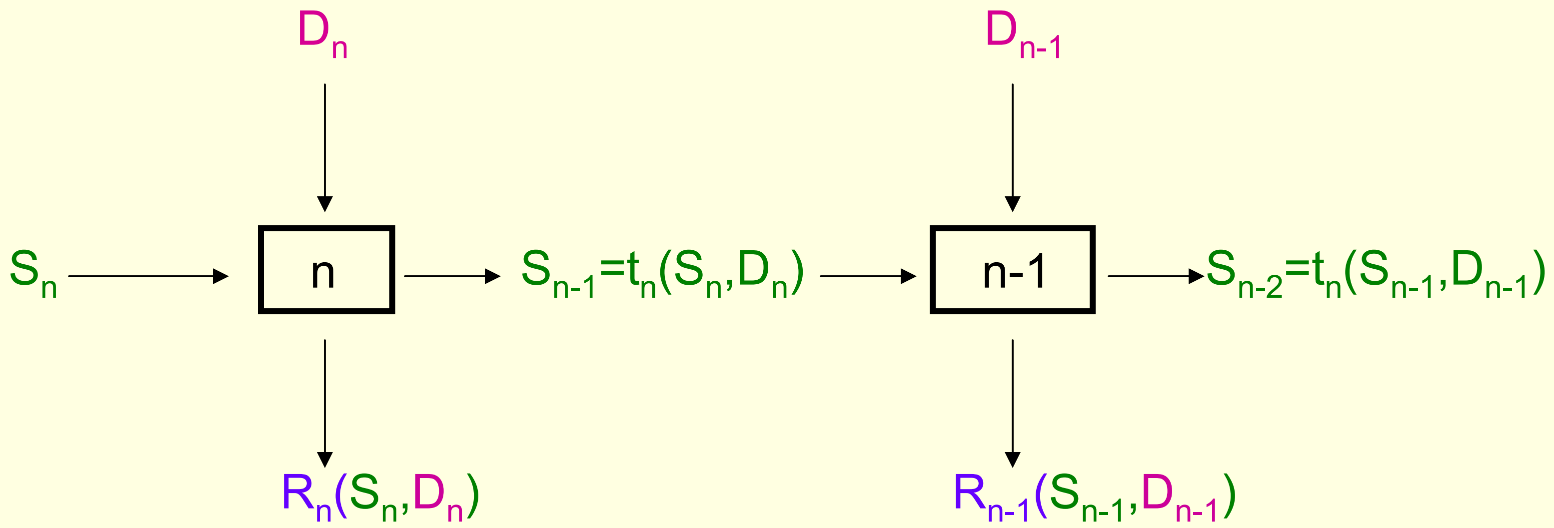
- Optimal control theory

  \- Based on Pontryagin (Russian Mathematician) maximization principle

- Dynamic Programming

  \- Based on Bellman Principle of Optimality

Components of a DP problem



D_n : Decision at stage n

S_n : State at stage n

R_n : Return at stage n

$S_{n-1}=t_n(S_n,D_n)$: Transition function

Dynamic Programming

- Bellman's Principle of Optimality

" An optimal policy or a set of decisions has the property that what ever the initial sate and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision "

- Separation between stage return and the state transformation at stage *n* from the decisions and states at other stages

- Complete set of decisions are handled in a sequential way rather than a simultaneous way

- Based on the assumption of separability or Markovian assumption

DP vs LP

- advantages :
  - not restricted to linear relationships (non linear stage return and stated transformation functions)
  - efficient mean of solving problems with many stages
  - Can be readily extended accommodate problems in stochastic nature

Limitations

- Assumption of separabilty : the returns and the state transformation at a stage depend only on the decision made in that stage and the state in that stage

Application of DP to Natural Resources

Examples of Functions in Some Resource Control Problems

Natural resource	x_i	u_i	$g_i\{x_i\}$	$h_i\{x_i, u_i\}$	$a_i\{x_i, u_i\}^a$
Mine	Resource mass	Level of extraction	—	Level of extraction	Net returns from extraction = $p_i h_i - C_i\{x_i, u_i\}$
Irrigated crop	Crop biomass	Level of irrigation	—	Irrigation induced growth ($h_i < 0$)	Net returns from irrigation = $-p_n h_i (1+r)^{n-i} - c_i u_i$
Beef cattle	Liveweight	Level of feed	—	Weight gain ($h_i < 0$)	Net returns from feeding = $-p_n h_i (1+r)^{n-i} - c_i u_i$
Timber thinning	Timber biomass	Level of thinning	Autonomous growth	Level of thinning	Net returns from thinning = $(p_i - c_i) u_i$
Fishery	Fish biomass	Level of fishing effort	Autonomous growth	Level of harvesting	Net returns from harvesting = $p_i h_i - C_i\{x_i, u_i\}$

[a] p_i = price of product; c_i = cost per unit of control; $C_i\{\cdot\}$ = total control cost; r = rate of discount.

Source: Kennedy . J.O.S. 1986

Representation of DP Problem

$$\underset{u_i \ldots u_n}{Max} \sum_{i=1}^{n} \alpha^{i-1} a_i \left\{ x_i , u_i \right\} + \alpha^n \, F \left\{ x_{n+1} \right\}$$

Subject to

$$x_1 = \overline{x}_1$$

$$x_{i+1} = t_i \left\{ x_i , u_i \right\} \qquad (i = 1, \ldots\ldots, n)$$

Where

Sequence of decision to be made : $u_i \ldots u_n$

Transformation function : $x_{i+1} = x_i + g_i\{x_i\} - h_i\{x_i, u_i\}$

Final value associated with the terminal state : $F\{x_{n+1}\}$

Autonomous addition to resource stock between stage *i* stage *i+1 :* $g_i\{x_i\}$

Control reduction in the resource stock between stage *i* stage *i+1 :* $h_i\{x_i, u_i\}$

Stage return $a_i\{x_i, u_i\}$

Corresponding Recursive Equation

$$v_i\left\{x_i\right\} = \underset{u_i}{Max}\left[\alpha^{i-1}a_i\left\{x_i,u_i\right\} + \alpha V_{i+1}\left\{t_i\left\{x_i,u_i\right\}\right\}\right]$$

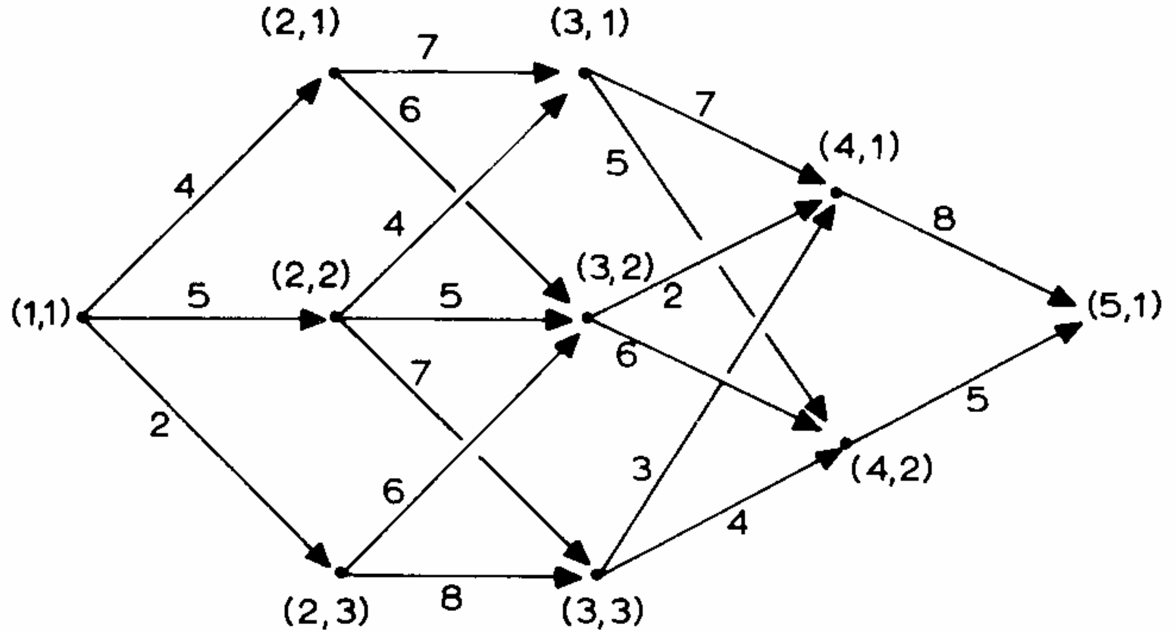
With

$$V_{n+1}\left\{x_{n+1}\right\} = F\left\{x_{n+1}\right\} \qquad (i = n,........,1)$$

$$x_1 = \overline{x}_1$$

Example : A Least cost network Problem

$$\min_{J_2, J_3, J_4} \sum_{i=1}^{4} a_i\{J_i, J_{i+1}\} + F\{J_4\}$$



A least-cost network problem.

$$\min_{J_2,J_3,J_4} \sum_{i=1}^{4} a_i\{J_i, J_{i+1}\} + F\{J_4\}$$

where $a_i\{J_i, J_{i+1}\}$ is the cost of linking (i, J_i) and $(i+1, J_{i+1})$; and

$$F\{J_4\} = \begin{cases} 8 & \text{if } J_4 = 1 \\ 5 & \text{if } J_4 = 2 \end{cases}$$

The recursive equation is

$$V_i\{J_i\} = \min_{J_{i+1}} [a_i\{J_i, J_{i+1}\} + V_{i+1}\{J_{i+1}\}] \qquad (i = 3 \text{ to } 1)$$

with

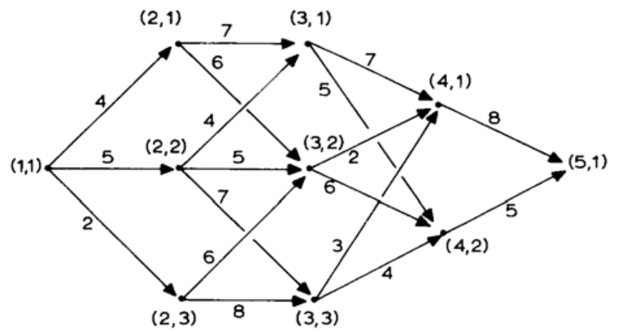
$$V_4\{J_4\} = F\{J_4\}$$

$$V_1\{J_1\} = \underset{J_2}{Min}\{a_1\{J_1, J_2\} + V_2\{J_2\}\}$$

$$V_2\{J_2\} = \underset{J_3}{Min}\{a_2\{J_2, J_3\} + V_3\{J_3\}\}$$

$$V_3\{J_3\} = \underset{J_4}{Min}\{a_3\{J_3, J_4\} + V_4\{J_4\}\}$$

Optimal path



A least-cost network problem.

Cost to (5,1) from Stage -3 Nodes

Node	Next Node (4,1)		(4,2)		Least cost V3{J3}	Optimal next mode
(3,1)	(7+8)	15	(5+5)	10*	10	**(4,2)**
(3,2)	(2+8)	10*	(6+5)	11	10	**(4,1)**
(3,3)	(3+8) .	11	(4+5)	9*	9	**(4,2)**

: Linking Nodes (3,1) to (4,1) , $a_3\{1,1\},7 + V_4\{1\},8 = 15$

: Linking Nodes (3,1) to (4,2) , $a_3\{1,2\},5 + V_4\{2\},5 = 10$

Optimal Path

Cost to (5,1) from Stage -2 Nodes

Node	Next Node						Least cost	Optimal
	(3,1)		**(3,2)**		**(3,3)**			next mode
							V2{J2}	
(2,1)	(7+10)	17	(6+10)	16*			16	**(3,2)**
(2,2)	(4+10)	14*	(5+10)	15	(7+9)	16	14	**(3,1)**
(2,3)			(6+10)	16*	(8+9)	17	16	**(3,2)**

Cost to (5,1) from Stage -1 Nodes

Node	Next Node						Least cost	Optimal
	(2,1)		**(2,2)**		**(2,3)**			next mode
							V1{J1}	
(1,1)	(4+16)	20	(5+14)	19	(2+16)	18*	18	**(2,3)**

Optimal Path

- (1,1) ⟶ (2,3) ⟶ (3,2) ⟶ (4,1) ⟶ (5,1)

  Minimum cost = 2 + 6 + 2 + 8=18

References

- Kennedy J.O.S. (1986)- *Dynamic programming , Application to Agriculture and Natural Resources*
- http://www.business.latrobe.edu.au/public/staffhp/jkennedy/
- Chiang A. C. (1992) - Elements of Dynamic Optimization

Optimal Irrigation : Kennedy (1986)

A farmer grows three horticultural crops in successive seasons over one year on 100 ha. Each crop takes four months, or one season, to reach maturity from the time of planting. The yield of each crop (in hundreds of tonnes per 100 ha) is given by

$$y_i = w_i - 0{\cdot}1 w_i^2 \qquad (i = 1 \text{ to } 3)$$

where w_i is the depth of water in centimetres received by the crop grown in the i-th season. The depth of water received depends on the height of water released from storage at the beginning of each season (u_i in metres) and rainfall received during each season (q_i in centimetres). The area of the dam is 1 ha, so

$$w_i = u_i + q_i$$

The dam is full at the beginning of the first season with a water height of 3 m. The amount of water which can be released at the beginning of any season is limited to integer values of metres of water, and by the amount in storage. Rainfall augments the water in storage. The catchment area is 100 ha, so 1 cm of rainfall raises the level of the dam (x_i) by 1 m, provided the dam is not full.

The farmer's objective is to select integer values of u_i so that the present value of receipts from sale of the crops is maximized. The price (in dollars per tonne) received for the i-th season crop is b_i. The problem can be formulated as:

$$\max_{u_1,u_2,u_3} \sum_{i=1}^{3} \alpha^{i-1} b_i(w_i - 0{\cdot}1 w_i^2)$$

subject to

$$0 \le u_i \le x_i \le 3, \quad u_i, x_i \text{ integer}$$

$$x_1 = 3$$

$$x_{i+1} = \min(x_i - u_i + q_i, 3)$$

with data

$$b = [50 \quad 100 \quad 150]$$

$$q = [2 \quad 1 \quad 1]$$

$$\alpha = 0{\cdot}95$$

The backward recursive equation used to solve this problem is

$$V_i\{x_i\} = \max_{0 \leqslant u_i \leqslant x_i} [b_i(w_i - 0{\cdot}1 w_i^2) + \alpha V_{i+1}\{x_i - u_i + q_i\}] \qquad (i = 3 \text{ to } 1) \tag{2.3}$$

with

$$w_i = u_i + q_i$$

$$V_4\{x_4\} = 0$$

Irrigation Problem

Optimal irrigation returns ($000) _ Satge3

Dam water Level	Water released (ha ,m)												V3(X3)	U*{3}
	U3	0		U3	1		U3	2		U3	3			
	q3	w3		q3	w3		q3	w3		q4	w3			
	1	1		1	2		1	3		1	4			
X3	Rev	V{X4}		Rev	V{X4}		Rev	V{X4}		Rev	V{X4}			
0	13.5	0.0	13.5										13.5	0
1	13.5	0.0	13.5	24.0	0.0	24.0							24.0	1
2	13.5	0.0	13.5	24.0	0.0	24.0	31.5	0.0	31.5				31.5	2
3	13.5	0.0	13.5	24.0	0.0	24.0	31.5	0.0	31.5	36.0	0.0	36.0	36.0	3