ARE 521 QUANTITATIVE TECHNIQUES

Duality in LP

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Duality in Linear Programming

- LP Problem consists of two forms
  - Primal
  - Dual
- Every primal problem there exists a corresponding dual problem
  - Primal Maximization → Dual Minimization
  - Primal Minimization → Dual Maximization

General LP Problem can be expressed as

$$Max\,(or\,Min)\,Z = \sum_{J=1}^{n} c_j x_j$$

Subject to

$$\sum_{J=1}^{n} a_{ij} x_j (\leq, =, \geq) b_i, (i = 1, 2, \ldots\ldots\ldots m)$$

$$x_j \geq 0, (j = 1, 2, \ldots\ldots\ldots n)$$

Primal and dual in Matrix form

- Primal

max $Z = c'x$

s. t.

$$Ax \le r$$

$$x \ge 0$$

Dual

min $Z^* = r'y$

s. t.

$$A'y \ge c$$

$$y \ge 0$$

Example : Max profit

Resource Requirement

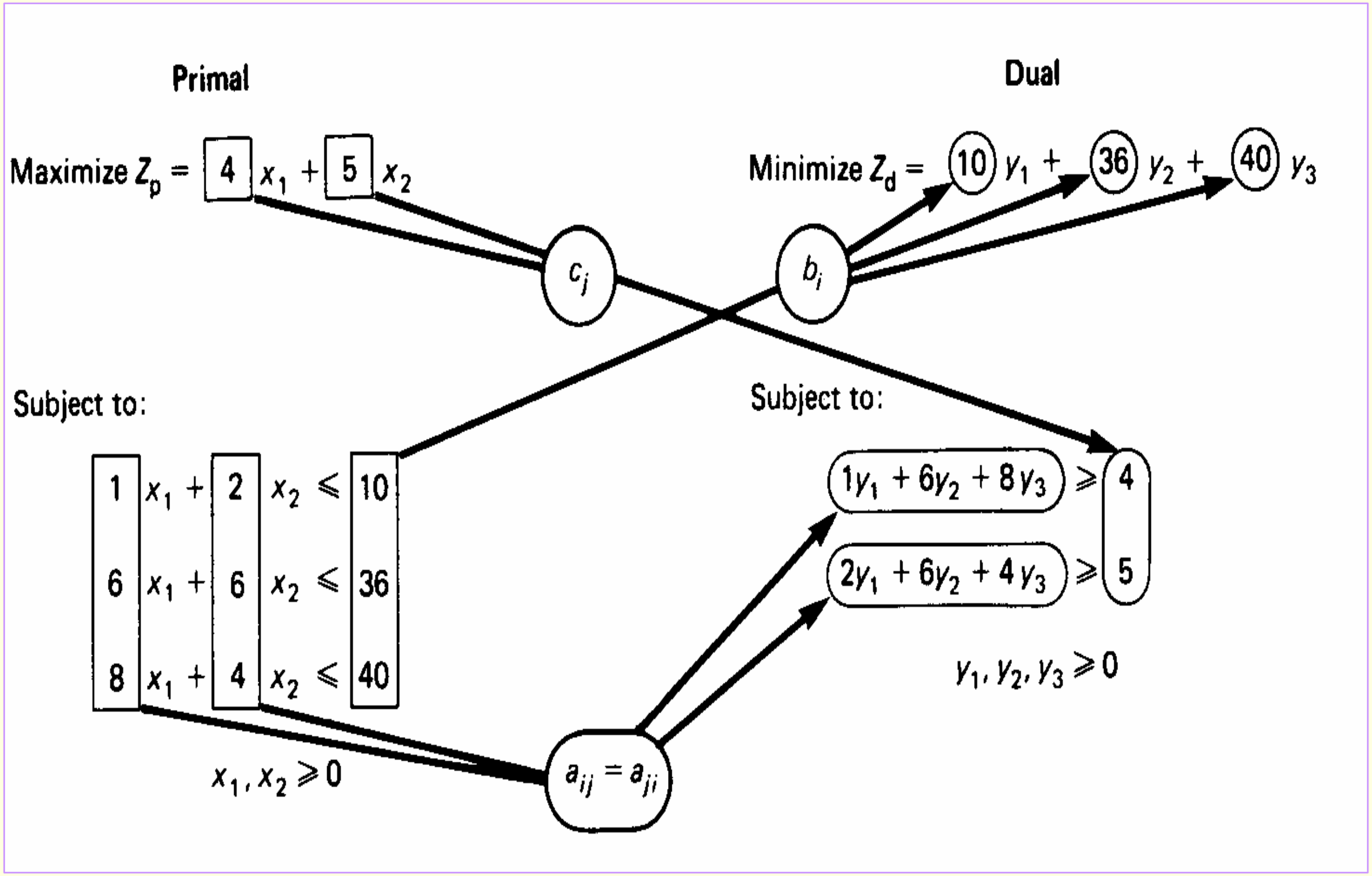
Resource	Product 1	Product 2	Total Resource
Labor (hr/unit)	1	2	10 hr
Material (lb/unit)	6	6	36 lb
Storage (ft^2/unit)	8	4	40 ft^2
Price ($/unit)	4	5	

Set up Primal for Max and Dual for Min problem

Dual Form

- y_1 = marginal value of 1 hr of labor (imputed price of labor)
  y_2 = marginal value of 1 lb of material
  y_3 = marginal value of 1 ft^2 of storage space
- $y_{1,}$ $y_{2,}$ $y_{3,}$ correspond to the resource constraints in the primal problem (m=3 variables in the model)
- Primal has *n* choice variable and *m* constraints and dual has *m* variables and *n* constraints
- Right hand side elements (b_i) in the primal correspond to coefficient of the objective function of the dual
- The a_{ij} constraint coefficients become a_{ji} in dual

Primal and Dual Relationship



Constraints

- $1y_1 + 6y_2 + 8y_3 \geq 4$

  : the value of labor, material and storage used in producing a unit of X1 should be greater than or equal to marginal revenue contribution of Product X_1 (4, price of x1)
- $2y_1 + 6y_2 + 4y_3 \geq 5$

  : $2^*{}_{\text{MV of labor}} + 6 *{}_{\text{MV of material}} + 4 *{}_{\text{MV of storage}} \geq 5$

Farm Planning Problem

	Constar level	Corn	Oats	Soy	Wheat	Con type	RHS
Activity level							
GM	92	30	10	40	12		
Land	4	1	1	1	1	≤	100
Mlab	1.5	0	1	0	0.5	≤	100
Jlab	3	1	0	2	0	≤	80

Max Z = 30x1 + 10x2 + 40x3 + 12x4 + 0s1 +0s2 + 0s3 (objective function)

subject to

1x1 + 1x2 + 1x3 + 1x4 + 1 s1 + 0 s2 +0 s3 = 100 --- (Land)

0x1 + 1x2 + 0x3 + .5x4 + 0 s1 + 1 s2 +0 s3 = 100 --- (March Labor)

1x1 + 0x2 + 2x3 + 0x4 + 0 s1 + 0s2 +1s3 = 80 --- (January Labor)

x1 , x2, x3, x4, s1 ,s2 , s3 ≥ 0

Optimal Tableau

Cj			30	10	40	12	0	0	0
Cb	Basis	bi*	x1	x2	x3	x4	s1	s2	s3
12	X4	20	0	1	-1	1	1	0	-1
0	S2	90	0	0.5	0.5	0	-0.5	1	0.5
30	x1	80	1	0	2	0	0	0	1
	Zj	2640	30	12	48	12	12	0	18
	Cj-Zj		0	-2	-8	0	-12	0	-18

Shadow prices

Opportunity cost

Optimal objective functional value

Primal and dual for the Farm Planning Problem

Primal

$$\text{Max } Z = 30x_1 + 10x_2 + 40x_3 + 12x_4$$

$$\begin{aligned} \text{S.T} \quad & x_1 + x_2 + x_3 + x_4 \le 100 \\ & x_2 + 0.5x_4 \le 100 \\ & x_1 + 2x_3 \le 80 \end{aligned}$$

$$x_1, x_2, x_3, x_4 \ge 0$$

Dual

$$\text{Min } C = 100Y_1 + 100Y_2 + 80Y_3$$

$$\begin{aligned} \text{S.T} \quad & Y_1 + Y_3 \ge 30 \\ & Y_1 + Y_2 \ge 10 \\ & Y_1 + 2Y_3 \ge 40 \\ & Y_1 + .5Y_2 \ge 12 \end{aligned}$$

$$Y_1, Y_2, Y_3, Y_4 \ge 0$$

Program for the Optimization problem in GAMS (PRIMAL)

```
Option limrow=10, limcol=10;

SET  J    crops /1, 2, 3, 4/
       I   resources /land, Mlab , Jlab/;

PARAMETERS   c(J) grossMar / 1 30, 2 10, 3 40, 4 12/
                    b(I)   limit  / land 100, Mlab 100, Jlab 80/;

    TABLE  a(i,j) activity matrix
         1     2     3      4
land     1     1     1      1
Mlab     0     1     0     0.5
Jlab     1     0     2      0 ;

VARIABLE TGM;
positive variables x;

EQUATIONS
 obj
 constrt;

Obj .. TGM =e= sum (j, c(j)* x(j));
constrt(i).. sum(j, a(i,j)* x(j)) =L= b(i);

model farm/all/;
solve farm using lp maximizing TGM;
```

```
                    S O L V E      S U M M A R Y

      MODEL    farm                    OBJECTIVE   TGM
      TYPE     LP                      DIRECTION   MAXIMIZE
      SOLVER   CPLEX                   FROM LINE   30

**** SOLVER STATUS          1 NORMAL COMPLETION
**** MODEL STATUS           1 OPTIMAL
**** OBJECTIVE VALUE                 2640.0000

                            LOWER     LEVEL     UPPER    MARGINAL

---- EQU obj                  .         .         .        1.000

---- EQU constrt

          LOWER     LEVEL     UPPER    MARGINAL

land      -INF    100.000   100.000    12.000
Mlab      -INF     10.000   100.000      .
Jlab      -INF     80.000    80.000    18.000

                            LOWER     LEVEL     UPPER    MARGINAL

---- VAR TGM                -INF    2640.000    +INF       .

---- VAR x

      LOWER     LEVEL     UPPER    MARGINAL

1       .      80.000     +INF       .
2       .        .        +INF     -2.000
3       .        .        +INF     -8.000
4       .      20.000     +INF       .
```

Shadow prices

Opportunity cost

Program for the Optimization problem in GAMS (DUAL)

```
Option limrow=10, limcol=10;

Set j crops /1,2,3,4/
    i resources /land, Mlab ,Jlab/;

Parameters c(j)   GM / 1 30, 2 10, 3 40, 4 12/
           b(i)   limit / land 100, Mlab 100, Jlab 80/;

Table a(j,i) activity matirx

        Land  Mlab  Jlab

1          1     0     1
2          1     1     0
3          1     0     2
4          1    0.5    0 ;


variable COST;
positive variables Y;

equations
obj
RESBAL;

Obj .. Cost =e= sum (i,b(i)* Y(i));
RESBAL(j).. sum(i,a(j,i)* Y(i)) =G= c(j);

model farmdual/all/;
solve farmdual using lp minimizing cost;
```

```
Optimal solution found.
Objective :        2640.000000


                      LOWER      LEVEL      UPPER
MARGINAL

---- EQU obj              .          .          .         1.000

---- EQU RESBAL

     LOWER      LEVEL      UPPER     MARGINAL

1    30.000     30.000      +INF      80.000
2    10.000     12.000      +INF        .
3    40.000     48.000      +INF        .
4    12.000     12.000      +INF      20.000

                      LOWER      LEVEL      UPPER
MARGINAL

---- VAR COST          -INF   2640.000      +INF          .

---- VAR Y

     LOWER      LEVEL      UPPER     MARGINAL

y1      .        12.000     +INF        .
y2      .          .        +INF      90.000
y3      .        18.000     +INF        .
```

Either cost saved/increased by relaxing the constraint by one unit

Reduced cost : incremental cost of forcing y1 to be in the basis